CONTINUITY IN CONCRETE BUILDING FRAMES

Practical Analysis for Vertical Load and Wind Pressure

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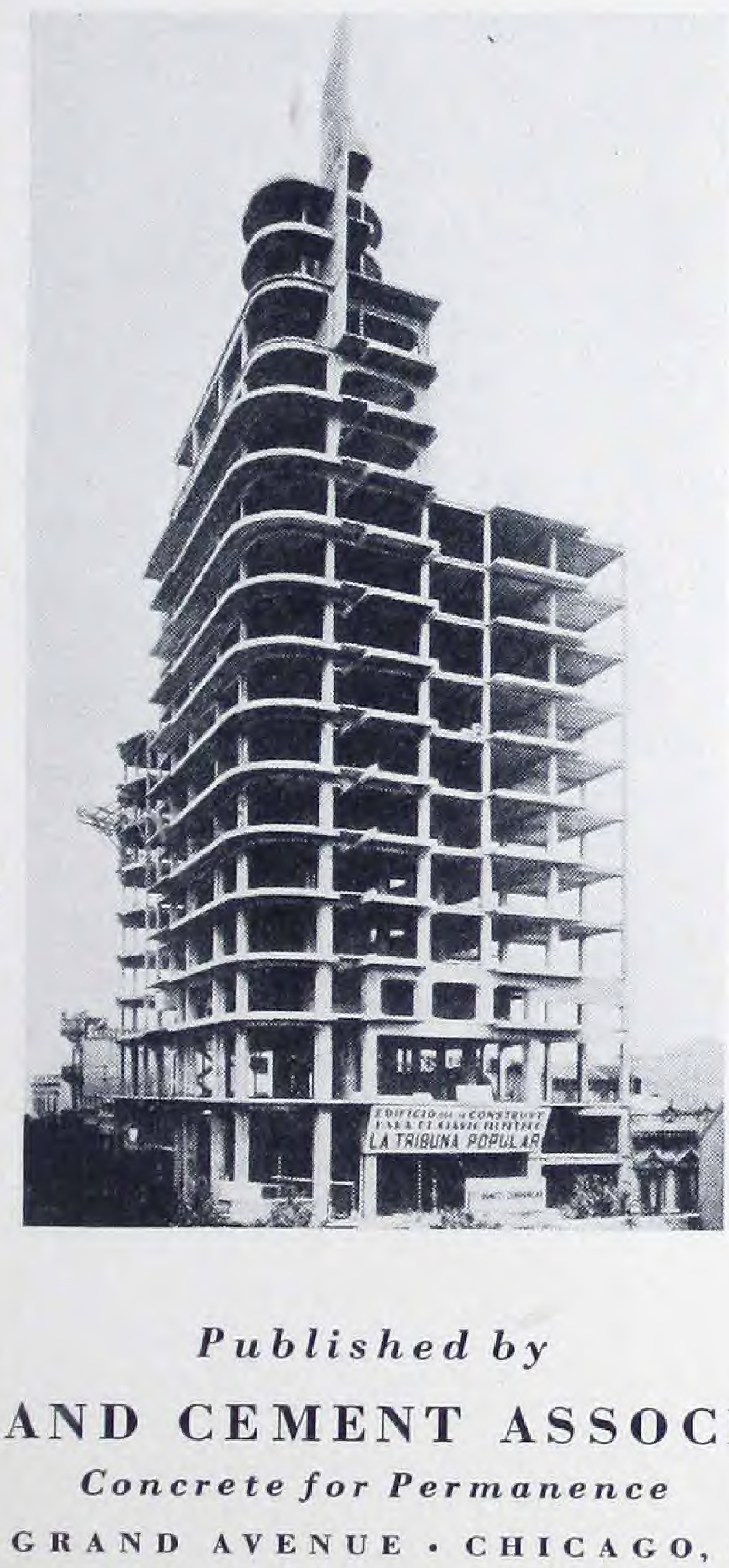
CONTINUITY IN CONCRETE BUILDING FRAMES



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PREFACE

The common practice of designing beams in building frames by use of moment coefficients and ignoring moments in columns is not always safe and adequate. It may result in distress in many structures and over-design in others. It disregards many important elements including concentrated loads, ratio of live load to dead load, ratio of span lengths in adjacent bays, ratio of column stiffness to beam stiffness, and ratio of column width to span length. The effect of all these elements upon design cannot be incorporated in three simple beam moments such as $^1/_{10}wl^2$, $^1/_{12}wl^2$ and $^1/_{16}wl^2$.

One reason why the moment coefficients are still used is that making a proper analysis has been a complex and lengthy task. This booklet is prepared to facilitate the analysis and to discuss quite fully the elements that go into the making of a well-balanced design. More specifically, the aim is to give constructive suggestions for improved practice and to present analytical procedures that are *practical*, i.e., quick, easy, convenient and yet sufficiently accurate for office design practice.

Considering the reduction in labor by the use of the proposed procedures, which are approximate, the accuracy of the results is surprisingly good. Even procedures that are mathematically exact do not yield exact results because they rest on assumptions that are approximate. In fact, better accuracy may be obtained from "approximate" analysis based on good assumptions than from "exact" analysis based on poor assumptions.

Good judgment and thorough understanding of the assumptions underlying the analysis are essential. For this reason, the presentation goes beyond the mere statement of analytical procedures, which are simple and occupy limited space only. Included are also: underlying assumptions, discussion and special studies, most of the latter in form of numerical problems. Derivations that are not essential to the application of the procedures have been deferred to the appendixes.

The problems are recommended for careful study because they make the reader familiar with the simplicity of the procedure, the accuracy obtainable and the effect of varying assumptions.

Procedures are given for both vertical load and wind pressure. The discussion of wind pressure applies to earthquake analysis also, if earthquake shocks are treated in the customary manner as static horizontal loading.

The analysis presented for vertical load is essentially Professor Cross' method of moment distribution applied—with slight modifications—to two joints only. For very irregular cases, such as offset columns supported on transfer girders, the procedure may have to be extended by using moment distribution applied to more than two joints. Also for wind pressure, the highly irregular cases need special study beyond that given in this text.

December, 1935

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NOTATION

M = moment.
M_{AB} = end moment at joint A of member AB.
M^F = fixed end moment.
U = unbalanced moment.
W = total load on span.
w = load per linear foot.
P = concentrated load or axial load.
l = span length.
h = story height.
I = moment of inertia.
K = I/l or I/h.
n = ratio of "K for column" to "K for beam".
Q = a joint coefficient.
e = eccentricity.
d = depth of section.
a = width of supporting member.
b = width of section.
A = area.
c = diameter of circle.
V = end shear.
v = relative shear.
Θ = angle of rotation at end of member.
E = modulus of elasticity.
f = a coefficient expressing degree of restraint.
R = angle of joint translation.

Continuity in Concrete Building Frames

Vertical Load

General Procedure of Analysis

1. Introduction

A procedure is presented for determination of end moments in horizontal members of continuous building frames. It has certain characteristics in common with current analytical methods, and the underlying principles in the derivation—given in Appendix A—are the same as in, for example, the slope-deflection and the moment-distribution methods. Current analytical methods are not superseded but merely adapted so that results with satisfactory degree of accuracy may be obtained by the least amount of work.

In analysis of continuous frames, the concept of fixed end moments is especially useful; they are the moments required to fix the ends of members, or—as is sometimes said—to lock the joints in frames. Fixed end moments, M^F, may be expressed in terms of the loading, W, and the span length, l; and M^F for one member is independent of other members. Moment coefficients for prismatic beams* with fixed ends and subject to nineteen types of loading are compiled in Fig. 1, in which the example in the lower right corner illustrates the use of the moment coefficients.

While determining values of M^F as, for example, in Fig. 2, joints C, A, B and D are assumed to be temporarily locked (prevented from rotating). If the artificial restraint is removed, each joint will rotate under the influence of the unbalanced moment: the difference between the fixed end moments. The use of unbalanced moments—denoted as U—is convenient in frame analysis, and U may readily be determined from values of M^F.

When joints rotate, new moments are induced at the ends of the members, and the final moment is equal to the sum of M^F and "induced moments." The problem to be solved by analysis of typical building frames is principally to determine the induced moments.

The induced moments are usually small compared with M^F. They may—in a sense—be considered as a correction to M^F, and a high degree of accuracy in the determination of induced moments is not essential. These moments will be approximated in the procedure which follows, and the work involved in the frame analysis will thereby be greatly reduced.

*Members will be assumed prismatic except in Section 14: Effect of Haunching.

Fig. 1*

Fixed End Moments—in Terms of *Wl*—for Prismatic Beams

2. *General Procedure of Analysis*

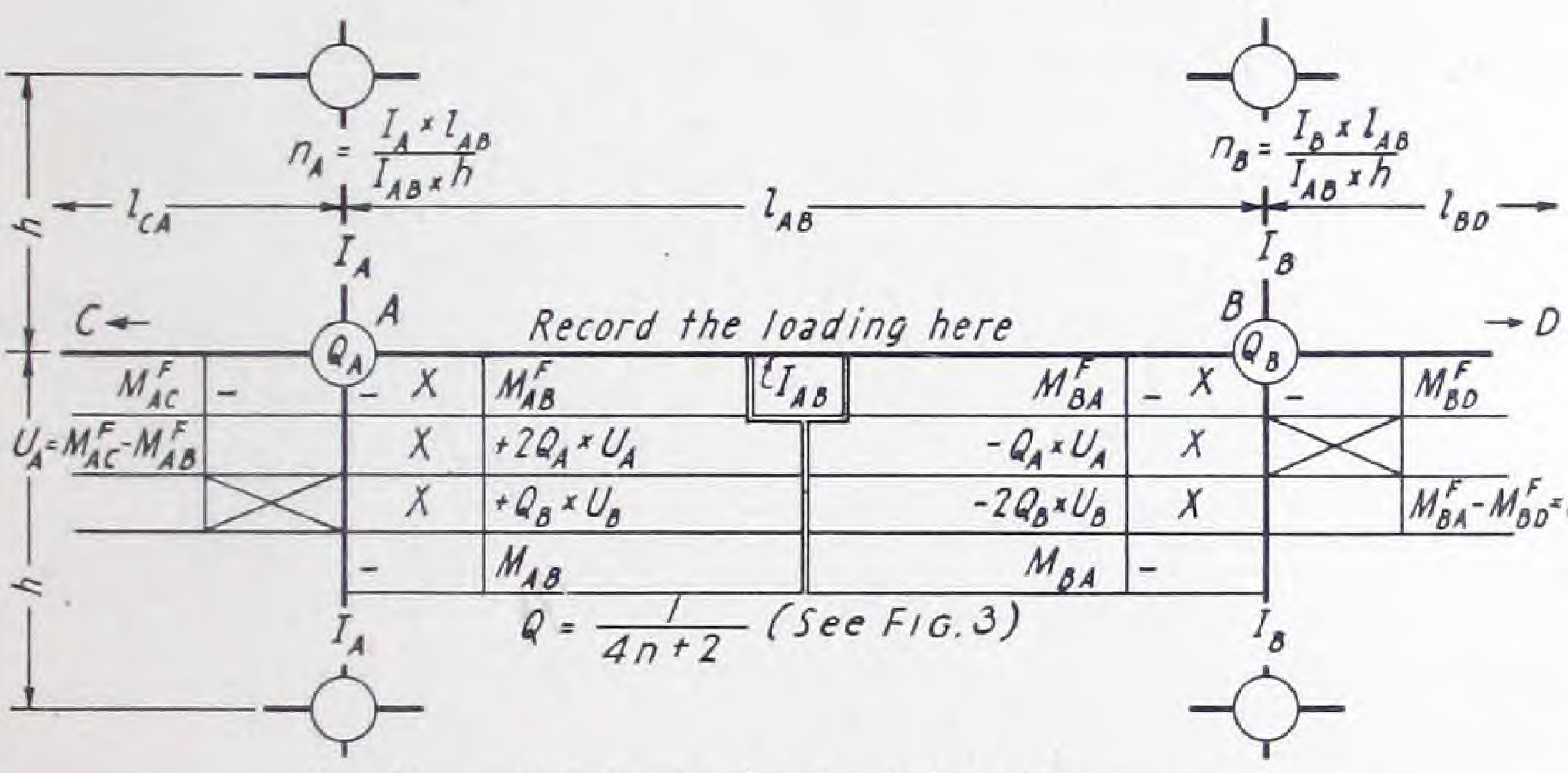
The value of M^F for prismatic members is a function of loads, W, and span lengths, l; but an induced moment depends also upon the moment of inertia, I, of the members in the frame. Let K denote the ratio of I/l for individual members, and n the ratio of K-value of a column to K-value

*By permission of the authors, Fig. 1 is taken—with slight modifications—from Fig. 5, page 85, in "Continuous Frames of Reinforced Concrete," by Cross and Morgan, published by John Wiley and Sons, 1932.

of a beam. Equations for n_A and n_B are given in Fig. 2. This figure illustrates the general procedure which is described in Steps (1) to (6):

(1) Record in Fig. 2 the loading on the three adjacent spans, l_{CA}, l_{AB} and l_{BD}.
(2) From this loading, determine values of M^F (using Fig. 1) and record M^F as negative quantities in Fig. 2.
(3) Compute the unbalanced moment at A, U_A, and at B, U_B, as M^F to the left minus M^F to the right of each joint. Record U with proper sign.
(4) Compute (or select, as explained in Section 3) values of n or Q (see Fig. 2 for formula for Q* applying to the case shown).
(5) Record products of U times Q as shown and with proper signs as in Fig. 2.
(6) Determine the final end moment as the algebraic sum of the three terms in the spaces marked X.

The calculations in Steps (1), (2) and (3) are those that precede most methods of analysis. Steps (4), (5) and (6) comprise the analytical work characteristic of the procedure in this text and will be discussed further in subsequent sections.



Negative moments give tension in top of beams.
I_A, I_B and h are average values for columns above and below joints A and B.
Note that U = moment to the left of joint minus moment to the right of joint.

Fig. 2

3. *Selection of Q-Values When Cross-Sections Are Not Known*

The relative values of K—denoted as n—must be chosen before a frame can be analyzed. How to select n, and the joint coefficient Q, is a question of the greatest importance to the designer.

In *reviewing* a frame already designed, the procedure is simply to compute K, n, Q and then proceed with the analysis. This is the type of problem

*See Appendix *A* for derivation.

usually treated in papers and textbooks. When only loads, spans and story heights are known, the values of Q must be selected.

The reader who is familiar with Professor Cross' moment distribution method* will recognize in the procedure in Fig. 2 the same characteristics as in that method: fixed end moments (M^F), unbalanced moments (U), carry-over factors, and distribution factors (Q). A number of cases were studied in most of which a Q-factor of $\frac{1}{4n+2}$ gave better results for frames with two joints than the corresponding factor, $\frac{1}{4n+4}$, used for regular moment distribution in frames with many joints. The equations $Q = \frac{1}{4n+2}$ proposed for joints in floors and $Q = \frac{1}{2n+2}$ proposed in Section 5 for joints in roofs are derived empirically consistent with the expressions for distribution factor known from the moment distribution method. A refinement of giving different Q-formulas for interior and exterior joints is deemed unwarranted.

According to the equation in Fig. 2, Q equals ½ when $n = 0$ (no column restraint) and decreases to zero with increasing column stiffness. The variation of Q between the limits of ½ and zero is plotted in Fig. 3.

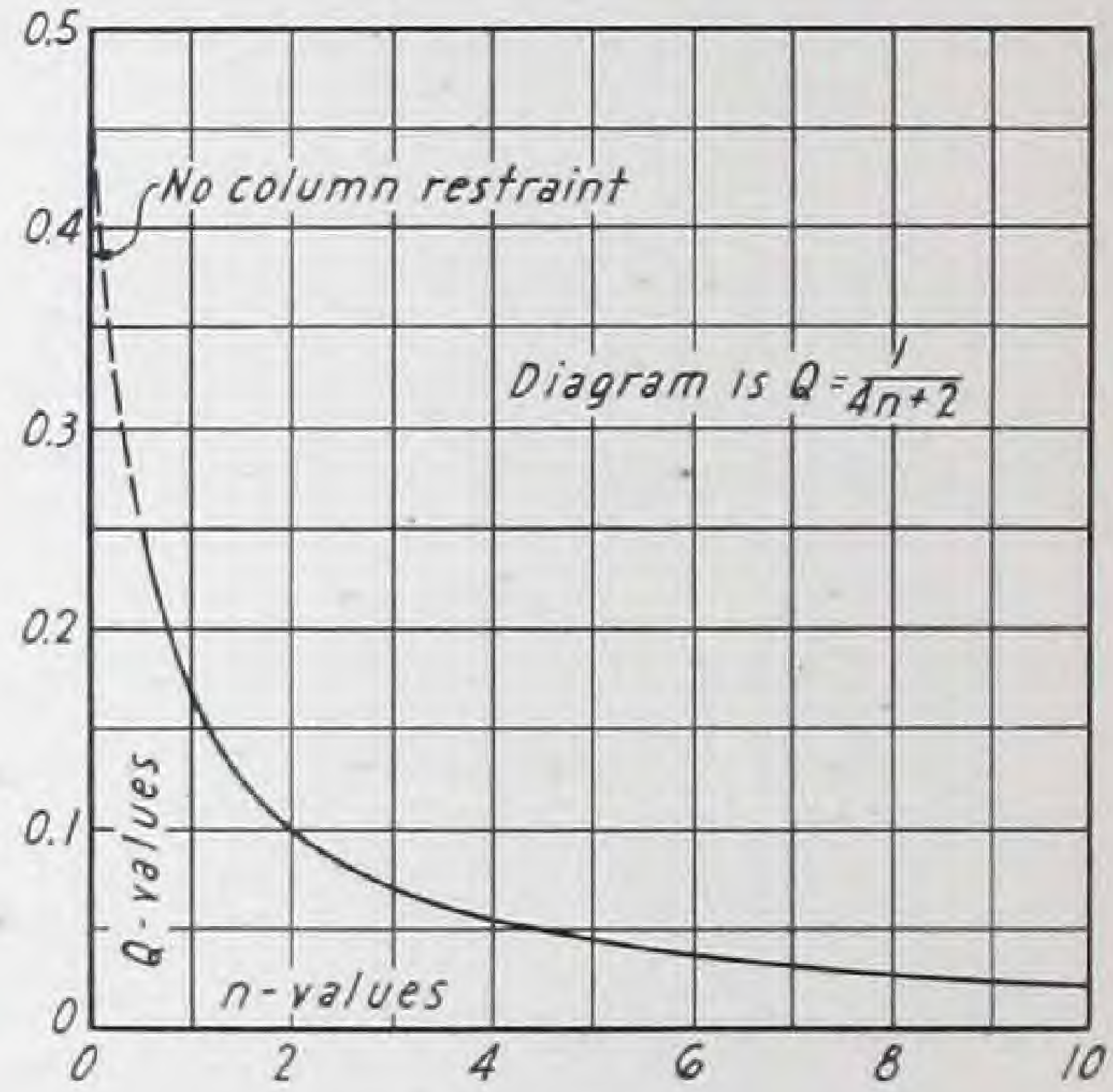


Fig. 3

When n is greater than about 4, the Q-values are small and vary only slightly. In this range, which includes the lower stories of tall buildings, $Q = \frac{1}{20}$ is a good average value.

Part of the Q-diagram near the ordinate axis ($n = 0$ in Fig. 3) corresponds to slightly restrained continuous girders. For ordinary building frames, n need seldom be taken smaller than one-half,** and the value of $Q = \frac{1}{4}$ may be selected as typical for frames with slender columns having n-values smaller than unity.

When n is between 1 and 4, the value of $Q = \frac{1}{8}$ may be considered typical. This is the range between the very slender and the very stubby columns.

If cross-sections are not known in ordinary building frames, the following suggestions may be used as a guide in selecting values of Q:

Values of n	Below 1	Between 1 and 4	Above 4
Column Classification	Slender	Medium	Stubby
Values of Q	¼	⅛	$\frac{1}{20}$

*See References 10 and 11.
**See Conclusions in Section 4.

4. *Examples of Analysis*

Problem 1. The frame in Fig. 4(a) has equal spans 20 ft. long and equal story heights; dead load equals 1000 lb. and live load 1200 lb. per lin. ft. Determine the maximum center moments in span AB and the corresponding end moment, M_{AB}.

The analysis will be carried out according to steps (1) to (6) on page 7 and arranged as in Fig. 2. Maximum center moment in AB is produced by placing live load in AB and also in alternate spans as in Fig. 4(a).* This determines the loadings and the fixed end moments, M^F. The beam and column dimensions are not known. Suppose the columns will be very slender and select $Q = \frac{1}{4}$ (see Section 3).

From the loading and span given, compute the fixed end moments at A and B by using the coefficient in case 5 in Fig. 1, which gives

$$M^F = -\tfrac{1}{12}\, W \times l = -\tfrac{1}{12}\, wl^2.$$

All fixed end moments are negative and are recorded in 1000 ft.lb. in the computation form below. The unbalanced moment, U, defined in Section 2 as the M^F to the left minus the M^F to the right, equals

$$(-33) - (-73) = +40 \text{ at } A, \text{ and}$$
$$(-73) - (-33) = -40 \text{ at } B.$$

The corrections to the fixed end moment due to the unbalanced moment of $+40$ at A equals

$$+2 \times QU = +2 \times \tfrac{1}{4}\,(+40) = +20 \text{ at } A, \text{ and}$$
$$-1 \times QU = -1 \times \tfrac{1}{4}\,(+40) = -10 \text{ at } B.$$

The unbalanced moment of -40 at B gives the following corrections:

$$+1 \times QU = +1 \times \tfrac{1}{4}\,(-40) = -10 \text{ at } A, \text{ and}$$
$$-2 \times QU = -2 \times \tfrac{1}{4}\,(-40) = +20 \text{ at } B.$$

The final moments, M_{AB} and M_{BA}, equal the algebraic sum of M^F and the two corrections.

$w=1000,\ l=20$	$n=\frac{1}{2}$	A $\frac{1}{4}$	$w=2200,\ l=20$		$n=\frac{1}{2}$	B $\frac{1}{4}$	$w=1000,\ l=20$
$-\frac{1}{12}\times1000\times20^2$	-33	-73	$-\frac{1}{12}\times2200\times20^2$	$-\frac{1}{12}\times2200\times20^2$	-73	-33	$-\frac{1}{12}\times1000\times20^2$
$(-33)-(-73)$	$+40$	$+20$	$+2\times\frac{1}{4}\times(+40)$	$-\frac{1}{4}\times(+40)$	-10		
		-10	$+\frac{1}{4}\times(-40)$	$-2\times\frac{1}{4}\times(-40)$	$+20$	-40	$(-73)-(-33)$
		-63	M_{AB}	M_{BA}	-63		

Maximum positive moment in span AB is

$$+\tfrac{1}{8} \times 2200 \times 20^2 - 63{,}000 = +110{,}000 - 63{,}000 = +47{,}000 \text{ ft.lb.}$$

In this case, due to symmetry, the moment computations might have been simplified as follows:

*Maximum center moment requires (1) full live load on AB and (2) minimum restraint at joints A and B; minimum restraint from beams adjacent to A and B is obtained when they carry no live load; and minimum restraint from columns adjacent to joints A and B requires load on alternate spans as shown on floor above and below floor AB in Fig. 4(a).

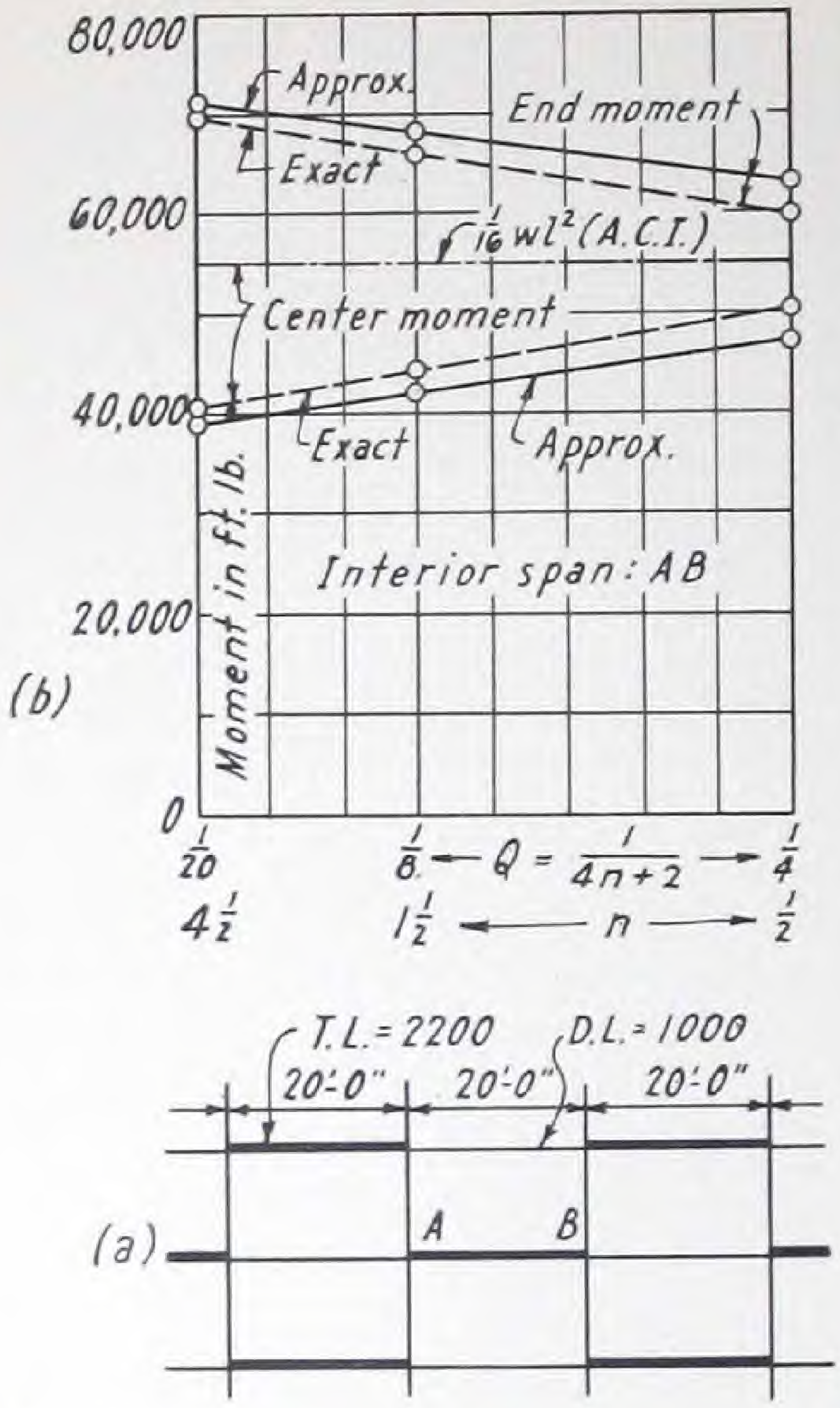


Fig. 4

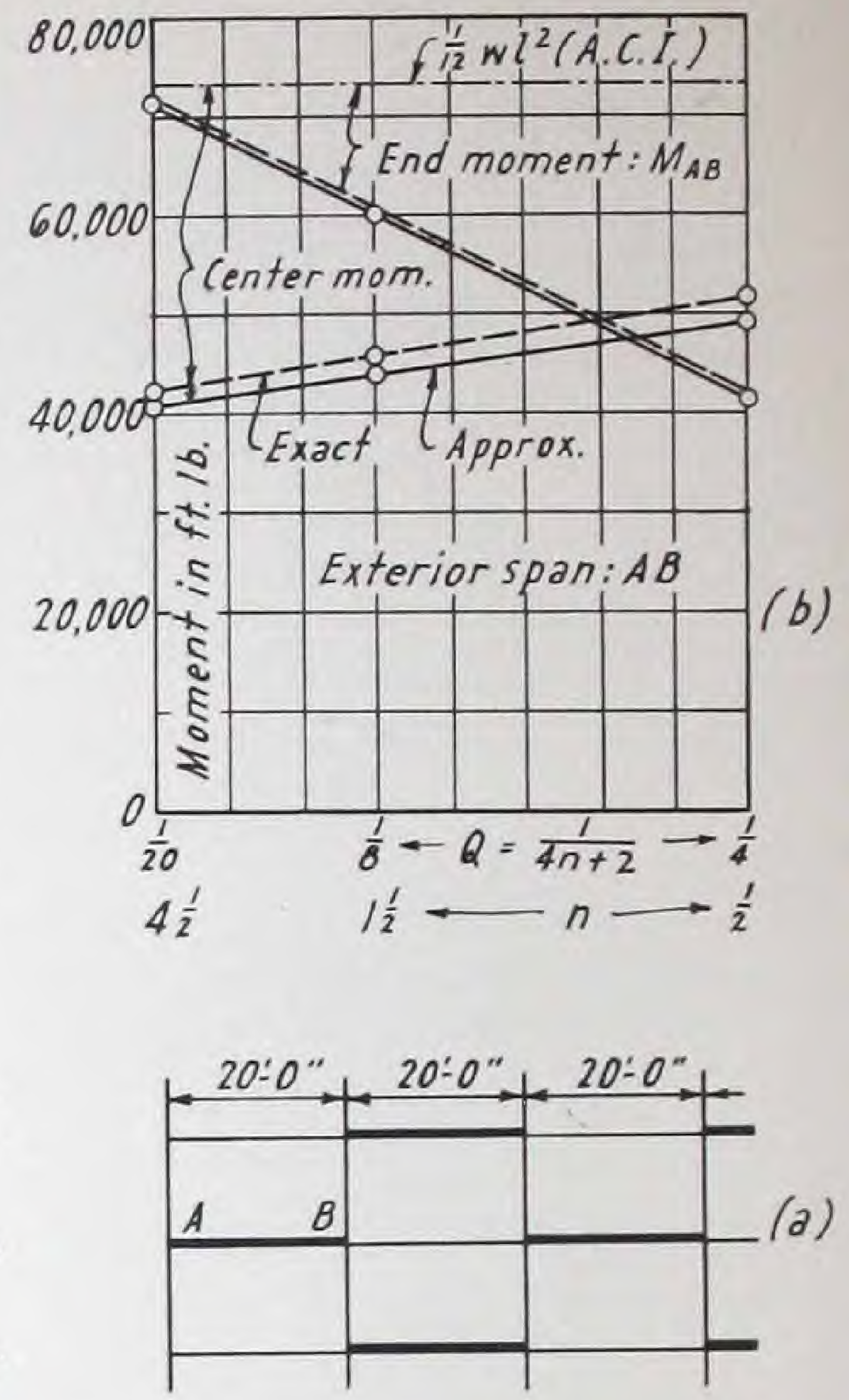


Fig. 5

$M_{AB} = M_{BA} = -73{,}000 + \frac{1}{4} \times 40{,}000 = -63{,}000$ ft.lb., and
Max. pos. mom. $= +110{,}000 - 63{,}000 = +47{,}000$ ft.lb.

To appraise the effect of varying Q-values, the end and center moments given below are determined by both approximate and exact analysis* for Q equal to $^1/_{20}$, $\frac{1}{8}$ and $\frac{1}{4}$, using the load arrangement in Fig. 4(a).

Q	$\frac{1}{4}$	$\frac{1}{8}$	$^1/_{20}$
n	$\frac{1}{2}$	$1\frac{1}{2}$	$4\frac{1}{2}$
End Moment . . . . (Approx.)	−63,000	−68,000	−71,000
End Moment . . . . (Exact)	−60,000	−65,300	−69,700
Center Moment . . . . (Approx.)	+47,000	+42,000	+39,000
Center Moment . . . . (Exact)	+50,000	+44,700	+40,300

The moments are plotted in Fig. 4(b). It is seen that increasing Q from $^1/_{20}$ to $\frac{1}{4}$ (equivalent to decreasing n from $4\frac{1}{2}$ to $\frac{1}{2}$), decreases the end moments and increases the center moment by about 20 per cent. The greatest discrepancy between any corresponding exact and approximate value is about 6 per cent or only one-third of the disparity in moment when Q varies from $^1/_{20}$ to $\frac{1}{4}$.

*"Exact Analysis" in this text indicates that the frame in its entirety was analyzed by the moment distribution procedure proposed by Professor Hardy Cross (see Reference 10). The intermediate calculations were recorded to the nearest 100 ft.lb., and slide rule was used throughout. Due to cumulative errors, the discrepancies in the "exact" values may possibly be as large as ±500 ft.lb.

The center moment computed by using the conventional coefficient ($\frac{1}{16}wl^2$) is also plotted and is about 17 per cent greater than the approximate moment determined for $Q = \frac{1}{4}$ and 41 per cent greater for $Q = \frac{1}{20}$.

In Figs. 4(a), 5(a) and 6(a), note that two joints and loads on spans adjacent to these two joints only are considered in the determination of beam moments by the proposed procedure. It is merely for comparison of these moments with the exact maximum moment values that the entire frames and all the loads shown in Figs. 4(a), 5(a) and 6(a) are included.

Problem 2. The frame in Fig. 5(a) has equal spans, 20 ft. long, and equal story heights; dead load equals 1000 lb. and live load 1200 lb. per lin. ft. Determine the maximum center moment in the exterior span AB and the corresponding end moment M_{AB}.

To produce the maximum value of the center moment, the loading is to be arranged as in Fig. 5(a).* Determine the fixed end moments accordingly. The beam and column dimensions are not known. Suppose the exterior columns are slender ($Q_A = \frac{1}{4}$) and the interior columns of average proportions ($Q_B = \frac{1}{8}$), see Section 3. The calculations are as follows, with moments given in 1000 ft.lb.

$w=0$	$n=\frac{1}{2}$, $\frac{1}{4}$	A	$w=2200,\ l=20$		$n=1\frac{1}{2}$, $\frac{1}{8}$	B	$w=1000,\ l=20$
	0	-73	$-\frac{1}{12}\times2200\times20^2$	$-\frac{1}{12}\times2200\times20^2$	-73	-33	$-\frac{1}{12}\times1000\times20^2$
$0-(-73)$	$+73$	$+37$	$+2\times\frac{1}{4}\times(+73)$	$-\frac{1}{4}\times(+73)$	-18		
		-5	$+\frac{1}{8}\times(-40)$	$-2\times\frac{1}{8}\times(-40)$	$+10$	-40	$(-73)-(-33)$
		-41	M_{AB}	M_{BA}	-81		

The moment calculations in the above table may be written in the following simplified manner, the computation form in Fig. 2 being omitted:

$$\begin{array}{ll} \text{At } A: & -73{,}000+2\times\frac{1}{4}\times73{,}000+1\times\frac{1}{8}\times(-40{,}000)=-41{,}000 \\ \text{At } B: & -73{,}000-1\times\frac{1}{4}\times73{,}000-2\times\frac{1}{8}\times(-40{,}000)=-81{,}000 \\ \hline \text{Average:} & \qquad\qquad\qquad\qquad\qquad\qquad\qquad -61{,}000 \end{array}$$

The maximum center moment in span AB equals

$$+\frac{1}{8}\times2200\times20^2-61{,}000=+49{,}000\text{ ft.lb.}$$

To appraise the effect of varying Q_A, values of end and center moments given below are determined by both approximate and exact analysis for Q_A equal to $\frac{1}{4}$, $\frac{1}{8}$ and $\frac{1}{20}$; Q_B remaining equal to $\frac{1}{8}$.

Q_A	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{20}$
n_A	$\frac{1}{2}$	$1\frac{1}{2}$	$4\frac{1}{2}$
End Moment. . . . . (Approx.)	$-41{,}000$	$-60{,}000$	$-71{,}000$
End Moment. . . . . (Exact)	$-41{,}800$	$-60{,}500$	$-71{,}800$
Center Moment. . . . . (Approx.)	$+49{,}000$	$+44{,}000$	$+41{,}000$
Center Moment. . . . . (Exact)	$+51{,}000$	$+46{,}000$	$+42{,}500$

*See footnote on page 9. Note that the maximum value of M_{AB} is not produced by the loading in Fig. 5(a), but by a loading as shown in Fig. 8(c).

The moments are plotted in Fig. 5(b). It is seen by comparing Fig. 5(b) and Fig. 4(b) that the center moments are nearly identical in interior and in exterior spans; their variation when Q varies is relatively small. Compared with the conventional moment values of $^1/_{16}wl^2$ for interior and $^1/_{12}wl^2$ for exterior spans, the moments determined by the analyses will give a more economic design, especially in exterior spans.

The end moment at the exterior column, plotted in Fig. 5(b), is more sensitive to variations in the value of Q than any of the other moments plotted in Figs. 4 and 5. Values of Q should therefore be selected with greater care for exterior than for interior joints. The end moments at the exterior column determined by analysis are smaller than those obtained by the use of the conventional moment coefficient of $^1/_{12}$.

Problem 3. Frame dimensions and loading are the same as in Problems 1 and 2. Determine maximum end moments, M_{BA}, in span AB at first interior and at interior columns.

The loading arrangements required to produce the maximum moments are shown in Fig. 6(a).* The analyses that follow are carried out according to the procedure in Problems 1 and 2, assuming for all columns $Q = \frac{1}{4}$, $\frac{1}{8}$ and $^1/_{20}$. The results are tabulated below and plotted in Fig. 6(b). The end moment is slightly greater at the first interior column than at the other interior columns; and both moments are greater than the moments computed by using the conventional coefficient of $^1/_{12}$.

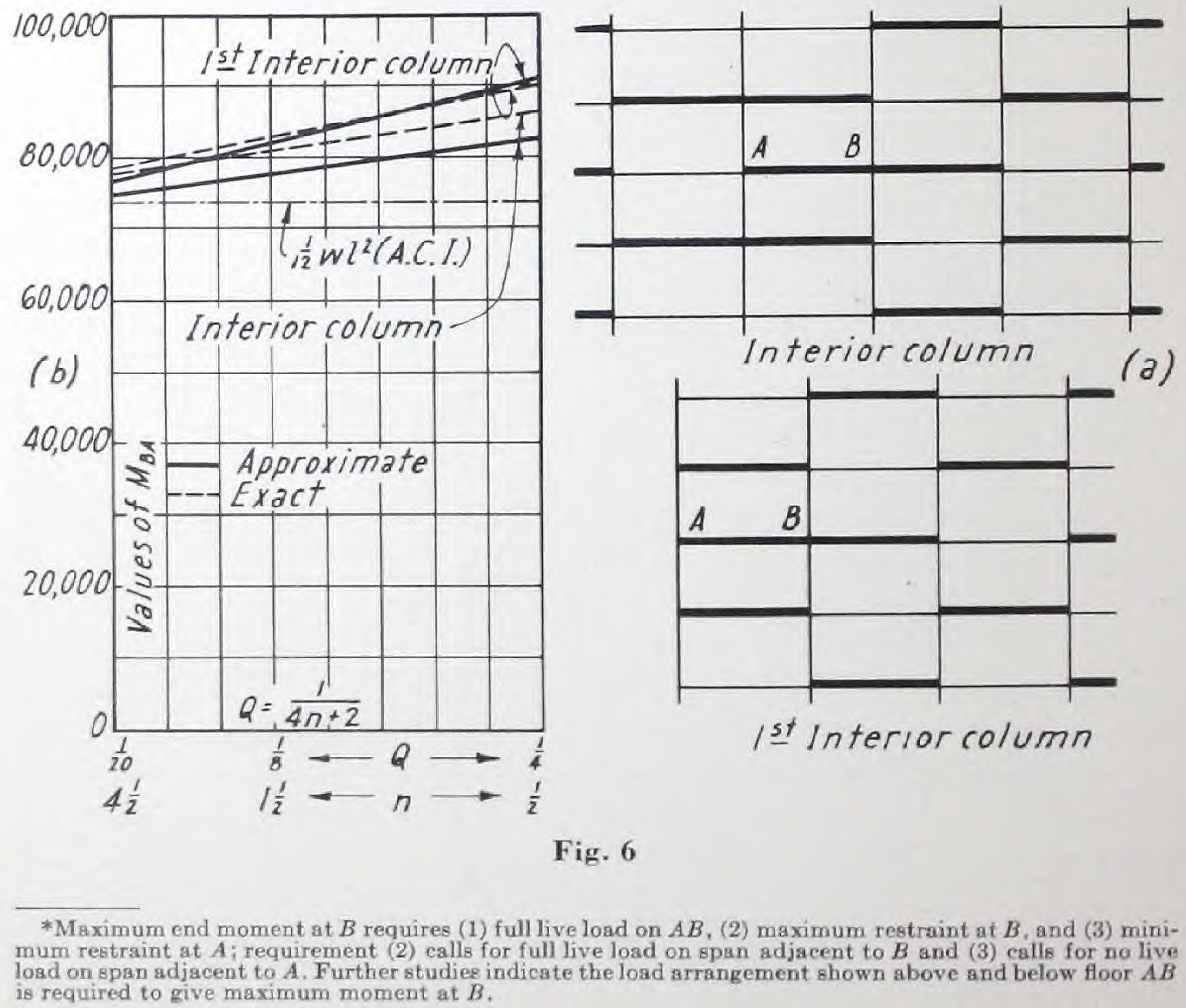


Fig. 6

*Maximum end moment at B requires (1) full live load on AB, (2) maximum restraint at B, and (3) minimum restraint at A; requirement (2) calls for full live load on span adjacent to B and (3) calls for no live load on span adjacent to A. Further studies indicate the load arrangement shown above and below floor AB is required to give maximum moment at B.

Q	¼	⅛	$^1/_{20}$
n	½	1½	4½
First Interior Column, End Moment. . . (Approx.) (Exact)	−91,000 −90,100	−82,000 −83,300	−77,000 −78,100
Interior Column, End Moment . . . . . (Approx.) (Exact)	−83,000 −86,500	−78,000 −81,700	−75,000 −77,600

Problem 4. The frame and loading will be similar to those in Problems 1, 2 and 3. The value of $Q = 1/8$ ($n = 1\frac{1}{2}$) will be used at all joints. The span in which moments are to be determined and also alternate spans will be 20 ft. long; for the other (intermediate) spans, lengths of 16 ft. and also 12 ft. will be chosen.

In the 20-ft. spans, maximum moments will be determined at the following points of the beams: Center Interior, Center Exterior, End Interior, End First Interior and End Exterior.

The loading arrangements required to produce the maximum moments are similar to those shown in Figs. 4, 5 and 6, except that the approximate end moment in the beam at exterior columns is produced by a loading as in Fig. 8(c) and determined by equation (6). All other approximate moments are determined according to the procedure in Problems 1 and 2, and the results* are summarized in Table 1.

Main Span: 20 ft. $Q = 1/8,\ n = 1\frac{1}{2}$	Adjacent Span Length		
Maximum Moments	20 ft.	16 ft.	12 ft.
Center, Interior. . . . . (Approx.) (Exact)	+42,000 +44,700	+43,000 +45,000	+44,000 +45,200
Center, Exterior . . . . (Approx.) (Exact)	+44,000 +46,000	+45,000 +46,100	+45,000 +46,200
End, Interior. . . . . (Approx.) (Exact)	−78,000 −81,700	−73,000 −77,000	−69,000 −72,900
End, First Interior. . . . . (Approx.) (Exact)	−82,000 −83,300	−76,000 −77,900	−70,000 −73,800
End, Exterior. . . . . (Approx.) (Exact)	−63,000 −65,000	−63,000 −65,400	−63,000 −65,800

Table 1

The agreement is good between results obtained by the approximate and the exact analyses. The change in moments in the 20-ft. spans is small when the length of the adjacent spans decreases. The maximum change in the center moments and in the end moment at the exterior column is insignificant. The end moments at the interior columns are somewhat more sensitive to change in length of adjacent spans.

Moments in the shorter spans will be discussed in Sections 8 and 9.

*The calculations may readily be duplicated by the reader by using Fig. 2.

Conclusions: For the cases investigated in Problems 1, 2, 3 and 4, the summaries in Figs. 4(b), 5(b), 6(b) and in Table 1 show that:

(1) The agreement between moments obtained by the approximate and by the exact procedure is satisfactory;

(2) The moments, except the end moments at exterior columns, change comparatively little when the relative stiffness, n, varies;*

(3) Moments obtained by using conventional coefficients are greater (up to about 75 per cent) than moments obtained by analysis, except that end moments at interior and first interior columns are smaller (up to about 20 per cent) than those obtained by analysis.

For concentrated loads which are symmetrical with respect to the midpoint of the span, the agreement between moments obtained by the proposed and by the exact procedure was found satisfactory. It is possible, however, to arrange loads so unsymmetrically with respect to the midpoints of spans that the proposed procedure may give unsatisfactory results; but such loadings are relatively infrequent.

It was recommended in Section 3 that the minimum value of n be chosen as one-half. If n is actually smaller than this value, using $n = \frac{1}{2}$ gives conservative results for the moments that are especially sensitive to variations of n; namely, the moments in columns and in beams at exterior columns (see Fig. 5(b) and Fig. 9). Other moments are affected far less by small changes in n; and, in general, n need not be taken smaller than one-half.

By the simplified procedure presented, maximum moments in continuous building frames may be determined by writing a few figures; and a more accurate and economical distribution of moments may be obtained than by the conventional moment coefficients.

Moments at critical points determined by the approximate method are generally found to be smaller than the exact moments. Consider, for example, the moments at interior column in Fig. 6(a). On the frame shown, live loads are placed on sixteen beams for exact maximum moment but on two beams only for approximate moment. The moment created by 2-span loading is about 4 per cent smaller than the exact moment. It should be noted, however, that the complete loading in Fig. 6(a) will exist rarely during the life of the structure. Since maximum moment will seldom be

*Within certain limits: (1) spans of nearly equal length, (2) loads uniformly distributed, and (3) live load nearly equal to dead load, the following proposed rules for determination of moments by use of coefficients will give results with a fair degree of accuracy (see Figs. 4, 5, 6 and Equations (3), (4), (6) in Section 7):

Center moments in all spans: $M = \frac{n}{21n - 2} wl^2$,

All end moments except at exterior columns: $M = \frac{3n}{34n - 2} wl^2$,

End moment at exterior columns: $M = \frac{n}{12n + 3} wl^2$,

End moment in exterior columns: $M = \frac{n}{24n + 6} wl^2$,

End moments in interior columns: $M = \frac{3n}{6n + 2} \times U$.

In certain instances, it will be justifiable to save work at the expense of accuracy by using for center moments the coefficient given instead of the procedure outlined in the problems. Further studies are desirable, however, before such coefficients can be recommended for general use.

reached, it is rational to use a smaller moment for design. The approximate moments, therefore, are good design values.

5. Moments in Beams in Roofs and Bottom Floors

One column connects with each joint in the roof, while there are two columns for each joint in floors. This difference will be reflected in the expression for Q in Fig. 2, for which the following equation will be substituted:

$$Q = \frac{1}{2n + 2}, \quad \ldots \ldots \ldots \ldots (1)$$

which applies to beams in roofs. Otherwise, the analytical procedure illustrated for floors and arranged as in Fig. 2 will apply to roofs also.

Beams in bottom floors may be analyzed as described and illustrated for regular floors using the value of

$$Q = \frac{1}{4n + 2}. \quad \ldots \ldots \ldots \ldots (2)$$

A part of the regular analytical work may be eliminated when the columns are relatively stubby. The calculations involving Q and U (see Fig. 2) may usually be omitted in the following cases: (1) at interior columns whose stiffness ratio n exceeds about 5, and (2) at exterior columns whose stiffness ratio n exceeds about $7\frac{1}{2}$. The fixed end moments may be used in these cases as final moments, since correction for joint rotation is unjustifiable.

6. Shear in Beams

Shear in a beam which is part of a frame is composed of the shear in the beam considered simply supported and a correction due to the end moments produced by the frame action.

Shear in a simply supported beam may be determined by statics. For concentrated loads, it is usually satisfactory to use the center-to-center distance between supports as the span in the shear calculations. When the load is uniformly distributed, the shear at the face of the support is one-half of the load on the clear span.

An end moment M_{AB}, creating tension in the top of the beam AB in Fig. 7(a) with span length l, produces reactions, R, the direction of which is

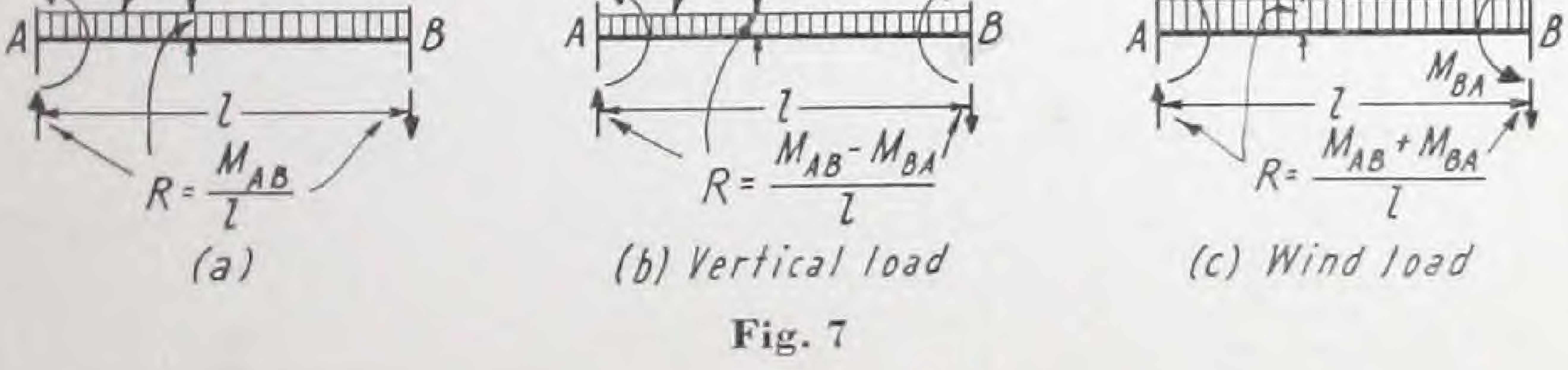


Fig. 7

upward at A and downward at B, the numerical value of the reactions being M_{AB}/l. The shear is constant and the shear curve is as shown in Fig. 7(a). The shear is also constant when moments are applied at *both* ends of the beam; but the numerator in the shear formula is the difference between the end moments when their direction is as shown in Fig. 7(b) and the sum of the end moments when their direction is as in Fig. 7(c). Allowance for sign of the shear in Fig. 7(b) may conveniently be made by use of the rule that the reaction on the beam is directed upward at the joint with the larger moment; the direction of the reactions in Fig. 7(a) and (c) may readily be determined.

To produce maximum end shear at joint B in span AB, see Fig. 6(a), it is required that (1) span AB is fully loaded, (2) M_{BA} is as large and (3) M_{AB} as small as possible. This indicates the loading shown in Fig. 6(a). For maximum end shear at exterior columns, loads may be arranged as in Fig. 5(a).

For illustration, the maximum shear in the end span at the centerline of the first interior column in Problem 3 for $n = \frac{1}{2}$ $(Q = \frac{1}{4})$ may be computed by the approximate procedure which follows, the loading being arranged as on the middle floor in Fig. 6(a), first interior column:

$$M_{BA} = -73{,}300 - 1 \times \tfrac{1}{4} \times 73{,}300$$

$$M_{AB} = -73{,}300 + 2 \times \tfrac{1}{4} \times 73{,}300,$$

from which

$$\frac{M_{AB} - M_{BA}}{l} = \frac{\frac{3}{4} \times 73{,}300}{20} = 2{,}800 \text{ lb.}$$

Shear at B equals $\frac{1}{2} \times 2{,}200 \times 20 + 2{,}800 = 22{,}000 + 2{,}800 = 24{,}800$ lb. In this case the simple beam shear, 22,000 lb., has been increased by $12\frac{1}{2}$ per cent at the first interior column and decreased by the same amount at the exterior column. One of the effects of continuity is to transfer shear; and it is often advisable to make allowance for shear transfer in the determination of column loads, particularly when the exterior columns are relatively slender.

7. *Column Moments and Beam Moments at Wall Columns*

Maximum moments in columns are produced by placing loads on several beams in at least two bays. The following convenient procedure, which is based on placing loads on few beams only, gives results that are satisfactory.

Fig. 8(a) shows a loading arrangement that produces, approximately, maximum moments in interior columns.* Live load is placed on all beams in one bay and there is no live load on beams in adjacent bays. The unbalanced moment, U, tends to rotate each joint through the angle Θ, and

*Maximum moment at the ends of a column requires that (1) the unbalanced moment at the ends is maximum, (2) the column deflection curve has reverse curvature, and (3) the rotation of the end joints is maximum. This indicates approximately the loading shown in Fig. 8. A somewhat larger moment would be produced if live load were placed as shown on two floors only and the loading reversed on the next floor above and below.

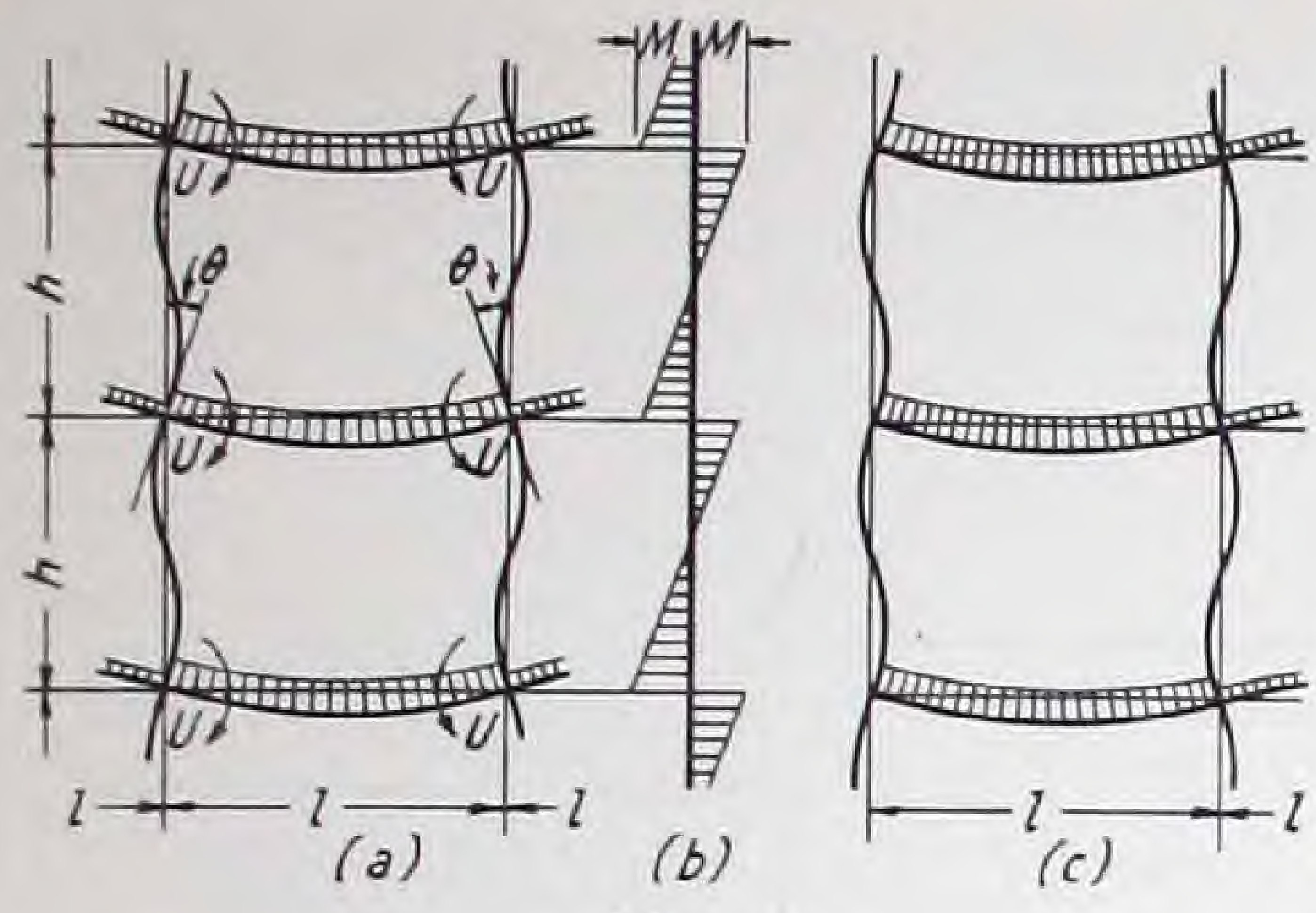


Fig. 8

the columns will deflect as indicated in Fig. 8(a). The moments, M, induced in the columns by U are shown in Fig. 8(b).

For the conditions in Fig. 8, it can be shown, see Section 28, that the end moments in the interior columns equal

$$M = \frac{3n}{6n + 2} \times U, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which

n = ratio of K for columns to K for beams (see expression for n in Fig. 9),
U = unbalanced moment due to the vertical loads indicated in Fig. 8(a).

In exterior columns, see Fig. 8(c), end moments may be computed from the following equation:

$$M = \frac{2n}{4n + 1} \times U, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which U equals the fixed end moment at the exterior joint.

The procedure is to

(1) determine U from the loading given,
(2) multiply U by the proper coefficient selected from Fig. 9.

The moments thus determined are approximate. To be conservative, it is advisable to choose large values of n and to adopt, say, $n = \frac{1}{2}$ as a minimum value.

The shear, V, in columns with reverse curvature as in Fig. 8 is the sum of the end moments divided by the story height, or in this case

$$V = \frac{2M}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since the end shears are opposite and nearly equal in the column above and the column below a joint, the beam between the columns gets little or no axial compression from the loading in Fig. 8.

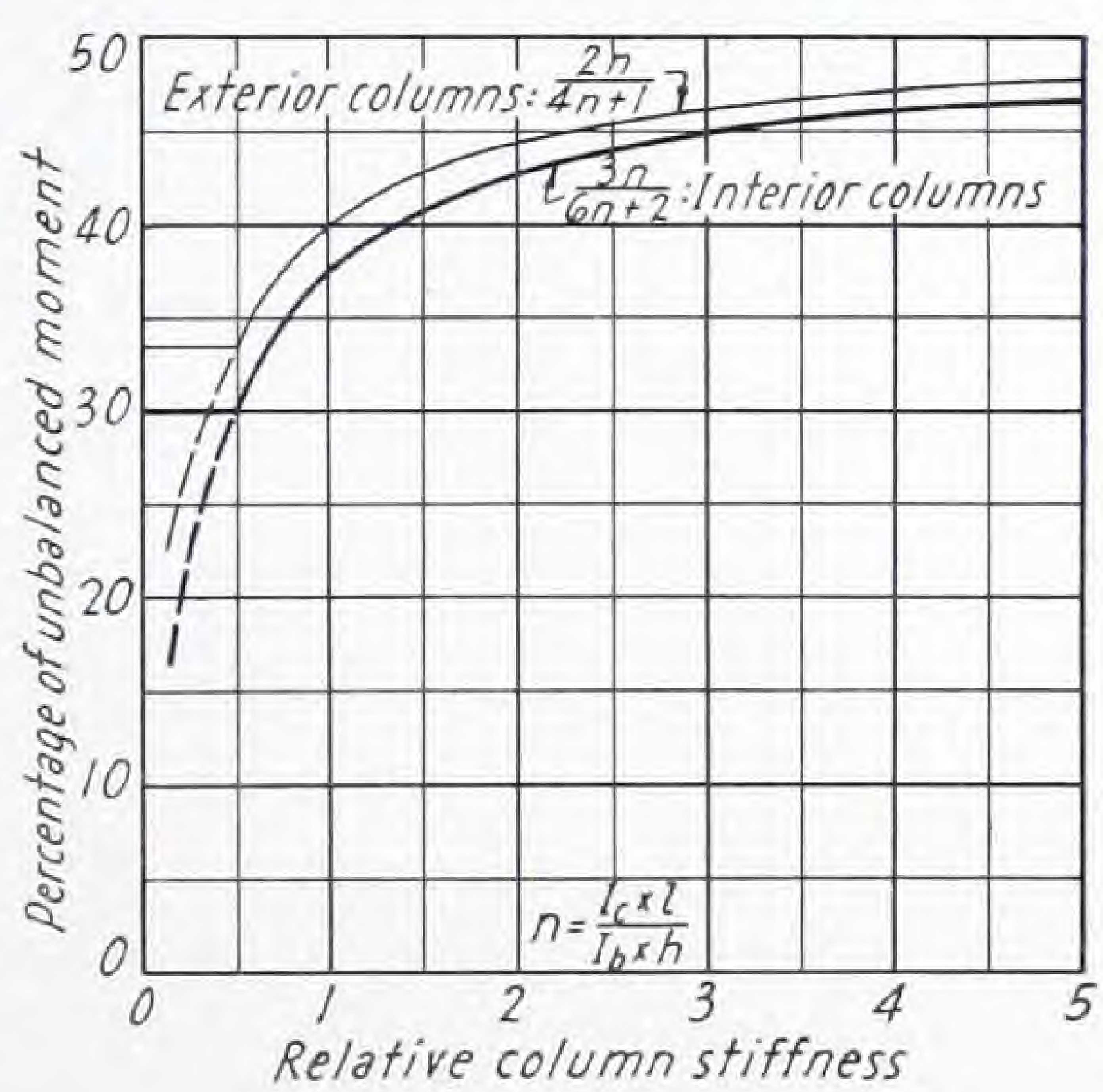


Fig. 9

Maximum end moment in beams at exterior columns equals the sum of maximum moments in the column above and the column below; it is therefore produced by a loading similar to that which gives maximum column moment and may be taken as

$$M_{AB} = \frac{4n}{4n + 1} \times U, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which U is the fixed end moment, M^F_{AB}.

Problem 5. A frame has spans and loading as in Problems 1 to 3; values of ½, 1½, 4½ are assumed for n. Determine maximum moments in the columns and in the beam at the exterior columns. For the loading arrangement in Figs. 8(a) and 8(c), the unbalanced moment, U, is

at interior cols.: $\frac{1}{12} \times (2200 - 1000) \times 20^2 = 40{,}000$ ft.lb.

at exterior cols.: $\frac{1}{12} \times 2200 \times 20^2 = 73{,}300$ ft.lb.

Using equations (3) and (4), the following values are computed:

n		½	1½	4½
Interior Columns	(Coefficient) (Moment)	0.300 12,000	0.410 16,400	0.465 18,600
Exterior Columns.	(Coefficient) (Moment)	0.333 24,400	0.429 31,500	0.474 34,700

Stresses due to these column moments are illustrated in the following examples:

(a) Interior column, $n = 4\frac{1}{2}$; gross section $= d^2 = 24 \times 24$ in.; axial load, $P = 500{,}000$ lb. The eccentricity is

$$e = \frac{M}{P} = \frac{12 \times 18{,}600}{500{,}000} = 0.45 \text{ in.}$$

(b) Exterior column, $n = 1\frac{1}{2}$; gross section $= d^2 = 16 \times 16$ in.; axial load, $P = 200{,}000$ lb. The eccentricity is

$$e = \frac{M}{P} = \frac{12 \times 31{,}500}{200{,}000} = 1.9 \text{ in.}$$

In a rectangular, homogeneous section with depth d, the extreme fiber stress for load P with eccentricity e is $\left(1 + \frac{6e}{d}\right)$ times the stress produced by load P applied concentrically. The increase in stress is

$$\text{for (a): } \frac{6 \times 0.45}{24} = 0.11, \text{ or 11 per cent,}$$

$$\text{for (b): } \frac{6 \times 1.9}{16} = 0.71, \text{ or 71 per cent.}$$

In this example, the effect of moment is negligible in the interior column; but the moment in the exterior column should not be ignored.

The maximum moments in the beam at the exterior column are, using equation (6) for the approximate values:

n		½	1½	4½
Coefficient		0.667	0.857	0.948
Moment	(Approx.) (Exact)	−48,900 −49,900	−62,800 −64,700	−69,500 −70,500

There is good agreement between the approximate and the exact moments.

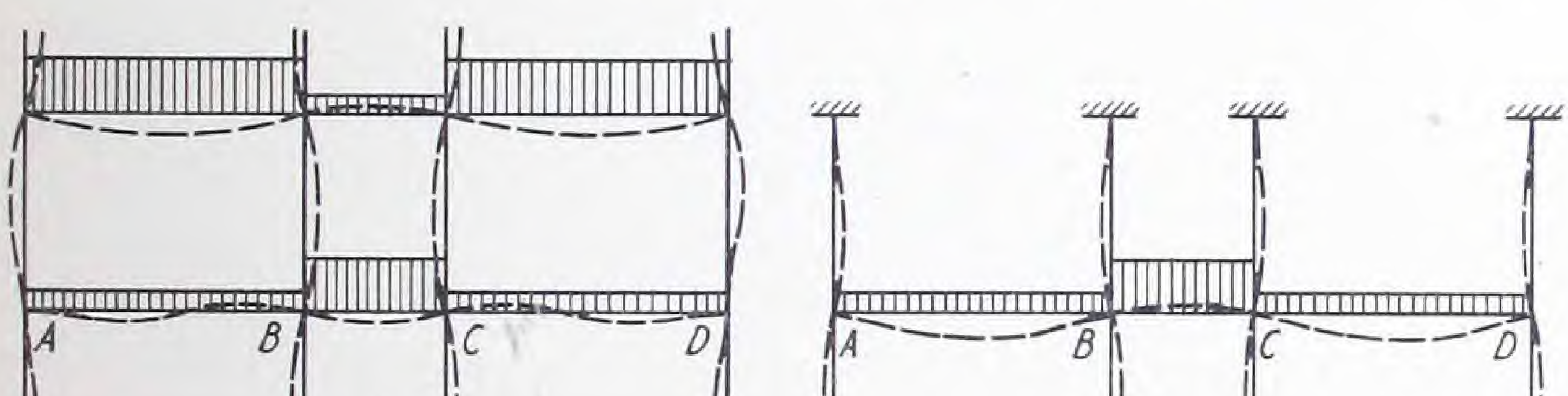
Special Procedure of Analysis

8. Assumptions Regarding Restraint at Far Ends of Columns

To obtain satisfactory results by the procedure in Fig. 2, the rotation must be relatively small at joints in the floor above and below the floor considered. If the rotations are large, a considerable error may result from assuming that the far ends of the columns are fixed or hinged. The errors that may be large are in the moments that depend upon *live load being placed in short spans which are flanked by long spans without live load, this loading distribution being reversed in the floor above and below.* In general, such moments are relatively small and not important; but a discussion of the problems involved is desirable and will be given in the numerical problems that follow.

Problem 6. The symmetrical frame in Fig. 10 has three bays, the span lengths being 20 ft. for the outer spans and 10 ft. for the center span. The dead load is 1200 and the live load 2400 lb. per lin. ft. All K-values will be assumed equal, and the value of n is therefore equal to unity. In order to produce (1) minimum center moment in AB, and (2) maximum center moment in BC, live load is placed on the short beam BC, the general loading being arranged as in Fig. 10(a).

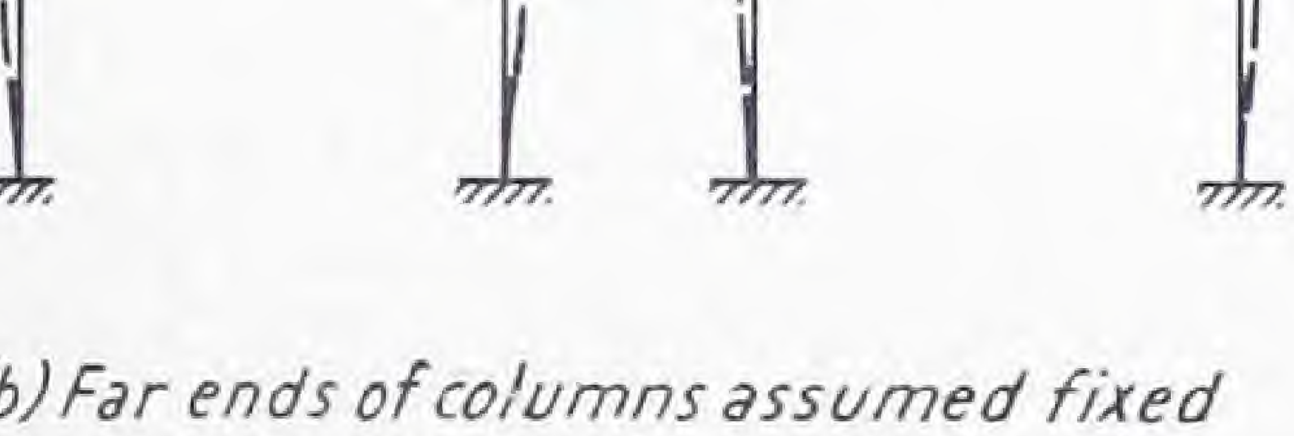
The actual condition of continuity is shown in Fig. 10(a), and an assumed condition with the far ends of the columns fixed is illustrated in Fig. 10(b).



(a) Actual condition of continuity

(b) Far ends of columns assumed fixed

	A		B			C
1	-40.0	+20.0	-40.0	-30.0	+15.0	
2	-28.2	+24.8	-42.2	-32.2	+12.8	
3	-25.5	+25.9	-42.6	-33.7	+11.3	
4	-39.9	+16.5	-47.2	-25.1	+19.9	
5	-28.4	+24.1	-43.4	-31.6	+13.4	
6	-35.6	+17.7	-48.9	-25.5	+19.5	
	A	(c)	B			C



(d)

Fig. 10

The deflection of the centerlines is indicated by the dotted lines. It is seen that the columns deflect in opposite directions and that the rotations of joints B and C are opposite in the two figures; this indicates that the assumptions made in Fig. 10(b) are unsatisfactory.

Moments in the beams AB and BC calculated on basis of various assumptions are tabulated in Fig. 10(c). Line 1 gives the moments obtained when all joints A, B, C and D are fixed. When these joints are released and the far ends of the columns remain fixed, as indicated in Fig. 10(b), the moments are as in Line 2; Line 3 gives the moments obtained when the columns are assumed hinged at joints above and below $ABCD$. Compare the moments in 2 and 3 with the exact moments, which are given in 4, and with the moments in 5, which are obtained by the procedure in Fig. 2. The results recorded in lines 2, 3 and 5 are similar; and they are in error, obviously because the joint conditions assumed at the far ends of the columns are not correct.

For loading and frame conditions similar to those in Fig. 10(a), too great an error may result from assuming the far ends of the columns fixed, hinged or partly restrained. In such cases, proper allowance should be made for the effect of loading on the floor above and below the floor considered. The allowance, which is made in the moments recorded in Line 6, will be discussed in Section 9.

9. Correction for Rotation at Far Ends of Columns

In making allowance for rotation of joints above and below floor $ABCD$ in Fig. 10(a), the analytical procedure in Fig. 2 is useful but may be modified as illustrated in Problem 7.

Problem 7. The frame, loading and quantities sought are the same as in Problem 6. The calculations, given first without correction for joint rotation above and below $ABCD$, are recorded in Table 2, which is arranged as Fig. 2. The end moments determined in Table 2—M_{AB}, M_{BA} and M_{BC}—are those given in Line 5 in Fig. 10(c).

A $\frac{1}{6}$				B $\frac{1}{6}$	
-40.0	$-\frac{1}{12}\times1200\times20^2$	$-\frac{1}{12}\times1200\times20^2$	-40.0	-30.0	$-\frac{1}{12}\times3600\times10^2$
$+13.3$	$+2\times\frac{1}{6}\times(+\underline{40})$	$-1\times\frac{1}{6}\times(+\underline{40})$	-6.7	-3.3	$+2\times\frac{1}{6}(-\underline{10})$
-1.7	$+1\times\frac{1}{6}\times(-\underline{10})$	$-2\times\frac{1}{6}\times(-\underline{10})$	$+3.3$	$+1.7$	$+1\times\frac{1}{6}(+\underline{10})$
-28.4	M_{AB}	M_{BA}	-43.4	-31.6	M_{BC}

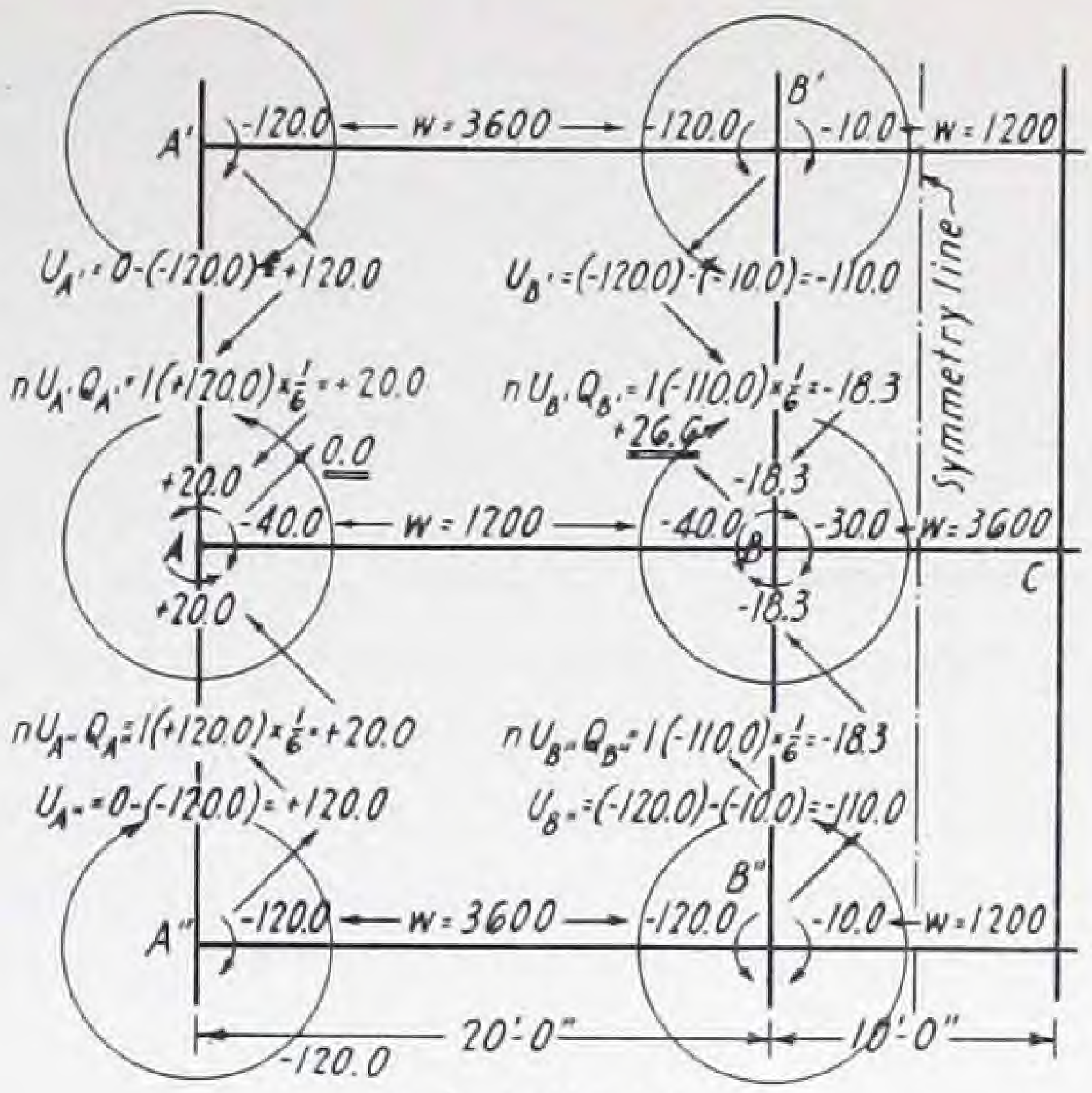
Table 2

The values of 40 and 10, underscored in Table 2, are the unbalanced moments at A, B and C. They are the only values which will be corrected; the procedure of correction is illustrated in the paragraphs which follow.

The numerical value of a moment at one end, A, of column AA' induced by a rotation due to an unbalanced moment, $U_{A'}$, at the other end, A', will be taken as $n \times U_{A'} \times Q_{A'}$ (see Fig. 11). The sum of moments origi-

nating at the joint above and the joint below A, equalling $nU_{A'}Q_{A'} + nU_{A''}Q_{A''}$, tends to rotate joint A in a direction opposite to the direction of $U_{A'}$ and $U_{A''}$. This sum will be included in the determination of the unbalanced moment at A, thereby introducing an allowance for rotation at A' and A''.

The procedure is illustrated in Fig. 11. The fixed end moment at A' and A'' is -120.0, and $U_{A'} = U_{A''} = 0 - (-120.0) = +120.0$. Choosing the value of $Q_{A'} = Q_{A''} = \frac{1}{6}$ and inserting $n = 1$ gives the sum of the moments induced in the column at A:



$Q = \frac{1}{6}$ at all joints

Fig. 11

$$1(+120.0)\tfrac{1}{6} + 1(+120.0)\tfrac{1}{6} = 2(+20.0),$$

the direction of which is *opposite* to the direction of M^F_{AB}. The moment tending to rotate A then equals

$$U_A = 0 - (-40.0) - 2(+20.0) = 0$$

Similarly at column $B'BB''$, taking $Q_{B'} = Q_{B''} = \frac{1}{6}$ gives

$$U_{B'} = U_{B''} = -120.0 - (-10.0) = -110.0, \text{ and}$$
$$nU_{B'}Q_{B'} + nU_{B''}Q_{B''} = 1(-110.0)\tfrac{1}{6} + 1(-110)\tfrac{1}{6} = 2(-18.3).$$

The direction of the moments induced in the columns is opposite to that of M^F_{BA}, which gives

$$U_B = -40.0 - (-30.0) - 2(-18.3) = +26.6.$$

In summary, to obtain a corrected value of U at any joint, take the M^F to the left minus the M^F to the right of the joint minus the moments induced in the columns above and below the joint. The moments induced at the near end of the columns may be taken as nQU written for the far end of the columns, U in this case also being equal to M^F to the left minus M^F to the right.

In the moment calculations in Table 2, replace $+\underline{40.0}$ by $\underline{0.0}$, $-\underline{10.0}$ by $+\underline{26.6}$, and $+\underline{10.0}$ by $-\underline{26.6}$. The calculations will then be as given in the following table:

A $\frac{1}{6}$				B $\frac{1}{6}$	
-40.0	$-\frac{1}{12}\times1200\times20^2$	$-\frac{1}{12}\times1200\times20^2$	-40.0	-30.0	$-\frac{1}{12}\times3600\times10^2$
0.0	$+2\times\frac{1}{6}\times(\underline{0})$	$-1\times\frac{1}{6}\times(\underline{0})$	0.0	$+8.9$	$+2\times\frac{1}{6}(+\underline{26.6})$
$+4.4$	$+1\times\frac{1}{6}\times(+\underline{26.6})$	$-2\times\frac{1}{6}\times(+\underline{26.6})$	-8.9	-4.4	$+1\times\frac{1}{6}\times(-\underline{26.6})$
-35.6	M_{AB}	M_{BA}	-48.9	-25.5	M_{BC}

The center moments equal:

in AB: $\frac{1}{8} \times 1200 \times 20^2 - \frac{1}{2}(35{,}600 + 48{,}900) = 17{,}700$ ft.lb.
in BC: $\frac{1}{8} \times 3600 \times 10^2 - 25{,}500 = 19{,}500$ ft.lb.

These moments are recorded in Line 6 of Fig. 10(c) and plotted in curve marked "6" in Fig. 10(d), in which they are compared with the exact moments (marked "4") and with the moments marked "2" which are obtained by assuming that the far ends of columns are fixed. It is seen by comparison of the moment curves in Fig. 10(d) that the assumption of fixed far ends is unsatisfactory in this case, and that the proposed procedure gives moments that are close to the exact moments.

It should be noted that for a frame as in Fig. 10(a), the general procedure in Fig. 2—without correction—gives good results for the moments in the floor above and below *ABCD*. For further illustration, the following calculations and comparisons of moments applying to the frame and loading as above are presented for two conditions: (1) all K-values equal to unity, $Q = 1/6$ (moments underscored); and (2) all K-values equal to unit, $Q = 1/6$, except for the intermediate beams, for which $K = 1/10$ and $Q = 1/42$ (moments given in parentheses). All moments are in ft.lb.

Maximum center moment in exterior span.

At exterior column:

$$-120{,}000 + 2 \times \tfrac{1}{6} \times 120{,}000 + 1 \times \tfrac{1}{6}(-110{,}000) = -98{,}300$$

At interior column:

$$-120{,}000 - 1 \times \tfrac{1}{6} \times 120{,}000 - 2 \times \tfrac{1}{6}(-110{,}000) = \underline{-103{,}300}$$

$$\text{Average:} = -100{,}800$$

At midpoint of span:

	Case (1)	Case (2)
Approximate analysis: $+180{,}000 - 100{,}800 =$	$\underline{+79{,}200}$	$(+79{,}200)$
Exact analysis:	$\underline{+82{,}000}$	$(+84{,}700)$

Minimum center moment in interior span.

	Case (1)	Case (2)
Approximate analysis: $+15{,}000 - 10{,}000 - \tfrac{1}{6} \times 110{,}000 =$	$\underline{-13{,}300}$	$(+2{,}400)$
Exact analysis:	$\underline{-15{,}600}$	$(+2{,}500)$

Maximum end moment in beam at exterior column.

	Case (1)	Case (2)
Equation (6): $-\dfrac{4 \times 1}{4 \times 1 + 1} \times 120{,}000 =$	$\underline{-96{,}000}$	$(-96{,}000)$
Exact analysis:	$\underline{-102{,}700}$	$(-103{,}800)$

The center moment of $-13{,}300$ ft.lb. under the heading "Minimum center moment in interior span" is produced by the loading shown on the top or the bottom floor in Fig. 10(a). For the same loading, the end moment in the interior span is

$$-10{,}000 - \tfrac{1}{6} \times 110{,}000 = -28{,}300 \text{ ft.lb.}$$

Since moments created by this loading are negative throughout the entire length of the center span, it will be necessary to detail top bars extending from span *AB* through *BC* into *CD*.

The practice of stopping top bars at or near the quarter-points of spans is safe where adjacent spans are nearly equal and live load is relatively small. When the difference between the lengths of adjacent spans increases

and/or the ratio of live to dead load increases, it becomes increasingly important to determine the points to which top bars should be extended, points which may be quite different from the quarter-points. It is recommended to arrange the load as for example on the top floor in Fig. 10(a), to compute the corresponding moments ($-13{,}300$ and $-28{,}300$ in the example), and to sketch the moment curve connecting these points. Such curves indicating minimum center moments are of importance in design and detailing of bar reinforcement.

In the cases illustrated, it is seen that (1) the accuracy obtained by use of Fig. 2 and equation (6) is satisfactory, and that (2) the change of K for the interior beams from unity to one-tenth has comparatively little effect upon the moments beyond the joints between which the change takes place.

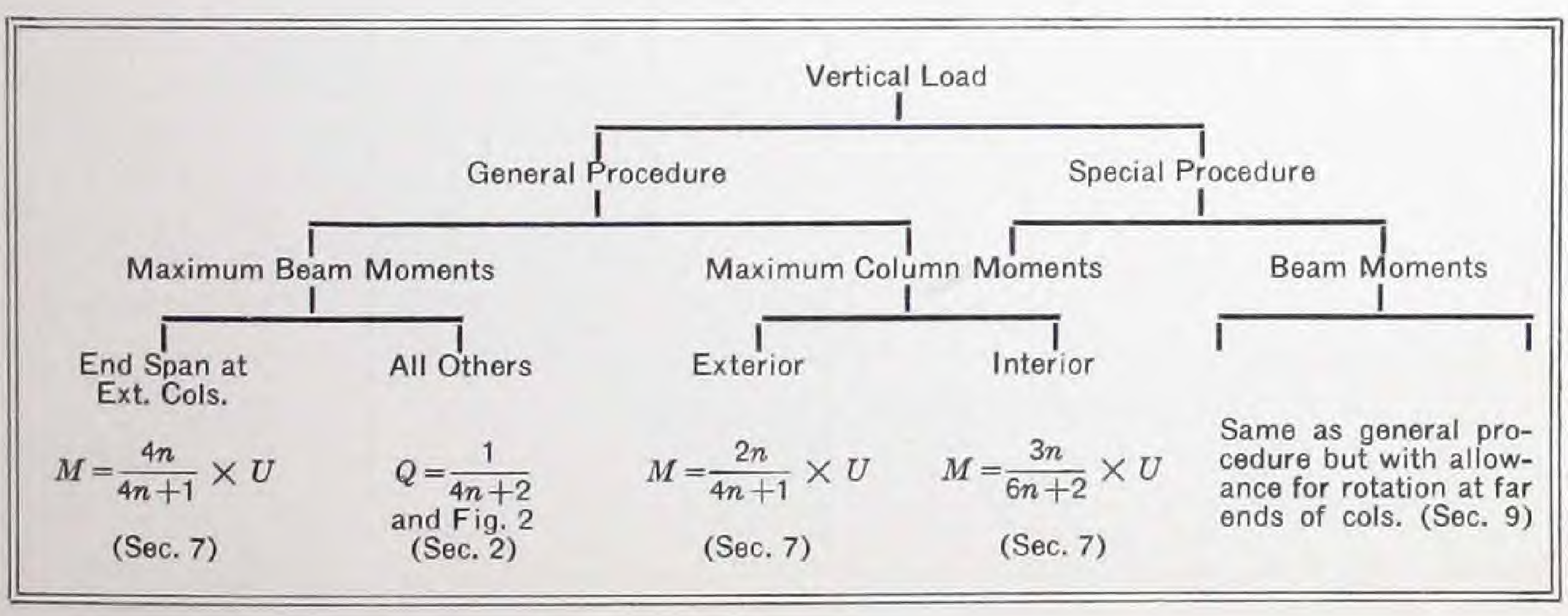
10. Summary

The following procedure is recommended for determination of moments produced by live load placed in short spans flanked by one or two long spans without live load. Arrange and carry out the calculations as in Fig. 2 with the exception that the unbalanced moments are modified in accordance with the following steps, (1) to (5), which are illustrated in Fig. 11.

(1) Determine Fixed End Moments (recorded to right and left of all joints).
(2) Compute Unbalanced Moments, U, at joints A', A'', B', B'' (on larger circles).
(3) Determine Q at joints A', A'', B', B''.
(4) Compute nQU at A', A'', B', B'' (recorded above and below A and B).
(5) Compute Modified Unbalanced Moment at A and B, to be used in Fig. 2, as M^F to the left minus M^F to the right, minus the moments induced in the column above and the column below each joint.

The calculations as arranged in Fig. 11 are simple and direct; they may usually be recorded without the schematic arrangement.

An outline of the analysis for vertical loading in accordance with the general and the special procedures is given below together with references to sections of the text where derivation and description are presented.



Discussion of Various Effects

11. Effect of Width of Columns

Analysis of frames deals with members which are assumed to be infinitely thin; but, actually, the members have finite widths that are often considerable as compared with span lengths and story heights. The discrepancy between actual and assumed condition gives rise to certain points of discussion, foremost among which is: Should the *centerlines* be chosen as the frame lines used in the analysis, or should other lines be chosen; for example, the lines whose spacings equal the *clear* lengths and heights?

Regardless of the choice of frame lines, the underlying principles and procedures of analysis remain unchanged. For example, the l-value in Fig. 2 may be chosen either as clear span or as centerline span; in both cases, the same procedure of analysis applies. It seems preferable to choose centerline distances since these are known at the beginning of an analysis, whereas sections and therefore the clear distances are unknown. Moments derived from analysis based upon centerline distances must be corrected, however, and this correction will be discussed.

Let the solid curve in Fig. 12 be the original moment curve as determined by analysis based upon centerline distances, l, from which is derived the negative end moment, M_c, and the positive center moment, M_p. Symmetry with respect to centerline of span has been assumed for simplicity in Fig. 12. Increasing the column width from zero to a will (1) lower the moment at b, (2) raise the moment at f, and (3) modify the moment curve in the general manner indicated by the dotted line in Fig. 12.

Let it be assumed that the moments corrected for effect of width of column will be

(a) $M_c - \frac{1}{6}Va$ at centerline of column
(b) $M_c - \frac{1}{3}Va$ at face of column and
(c) $M_p - \frac{1}{8}Va$ at midpoint of span,

in which V is the end shear at the centerline and a the column width. These moment corrections will be discussed and their consistency illustrated by

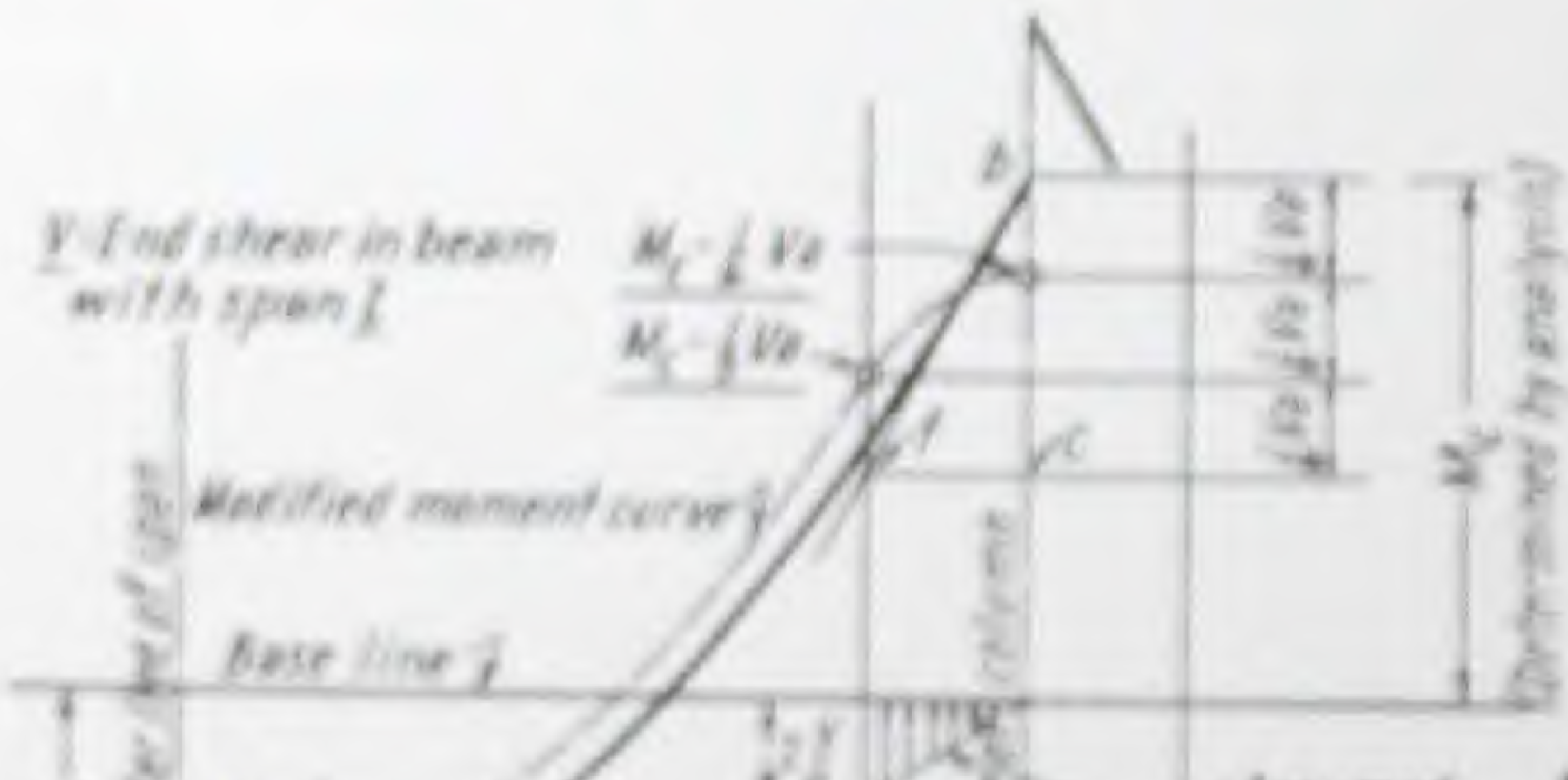


Fig. 12

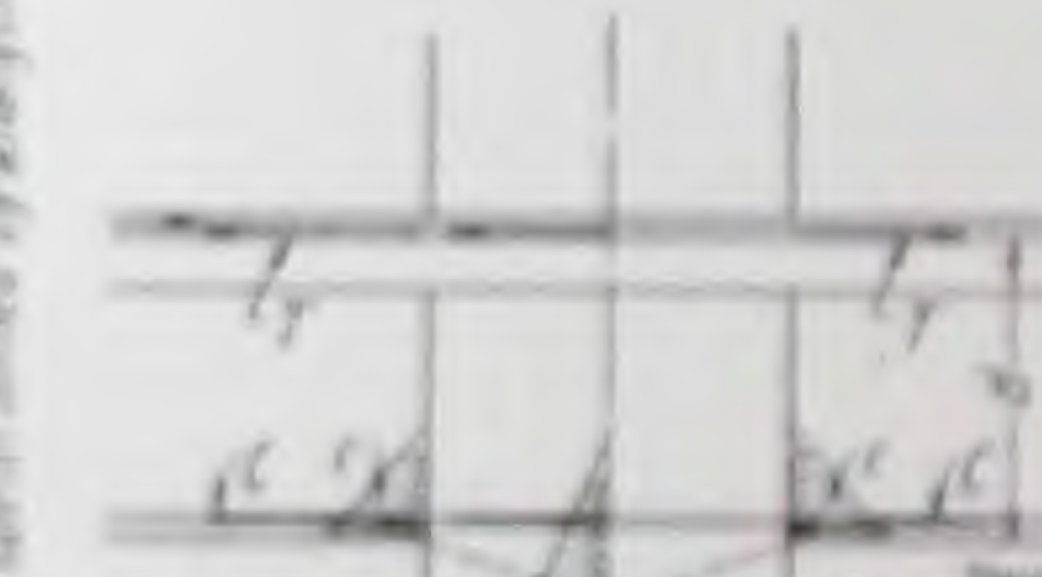


Fig. 13

comparison with moments obtained under certain typical conditions (a), (b) and (c), each case illustrating the corresponding point given above.

(a) Assume that the column reaction is triangularly distributed as shown in Fig. 12. The area of the triangle, or the reaction component, is $\frac{1}{2}\left(2\frac{V}{a}\right)\frac{a}{2} = \frac{1}{2}V$, and the distance from the center of gravity of the triangle to the column centerline is $\frac{2}{3}\left(\frac{a}{2}\right) = \frac{a}{3}$. The moment of the component with respect to the column centerline is $\frac{1}{2}V\left(\frac{a}{3}\right) = \frac{1}{6}Va$. This moment, which equals the reduction proposed in point (a), is approximately the amount to be deducted from the centerline moment, M_C, when a reaction concentrated in the column centerline is changed to the triangular reaction shown in Fig. 12.

(b) Consider two fixed end beams with the same load, w lb. per lin. ft., the span being l in one, but $(l - a)$ in the other beam. The difference between the end moments in these beams is

$$\frac{1}{12}wl^2 - \frac{1}{12}w(l - a)^2 = \frac{1}{6}wla - \frac{1}{12}wa^2.$$

The term $\frac{1}{12}wa^2$ is small compared with $\frac{1}{6}wla$; by disregarding it and substituting V for $\frac{1}{2}wl$, the difference between the two fixed end moments becomes $\frac{1}{3}Va$, which equals the correction recommended for the moment at face of column.

(c) The slope of the tangent, bf, to the moment curve at b in Fig. 12 equals V and the distance bc equals $V\left(\frac{a}{2}\right)$. The modified moment curve, which is shown dotted in Fig. 12, lies above the original moment curve between the column faces, the distance between the curves being somewhat less than $\frac{1}{3}V\left(\frac{a}{2}\right) = \frac{1}{6}Va$, say, $\frac{2}{3}(\frac{1}{6}Va) = \frac{1}{9}Va$. This is the reduction of the center moments recommended due to effect of width of column.

In Problem 1, the positive center moment, 47,000 ft.lb. was determined for $w = 2200$ lb. per lin. ft., and $l = 20$ ft. assumed to be the distance between the centerlines of columns. Making allowance for a column width of, say, 1 ft. 6 in., the corrected center moment is reduced to

$$47{,}000 - \frac{1}{9}\left(2200 \times \frac{20}{2}\right)1.5 = 47{,}000 - 3{,}700 = 43{,}300 \text{ ft.lb.}$$

The loading in Fig. 6(a) for maximum end moment, 91,000 ft.lb., at the first interior column, B, produces at B an end shear of $V = 24{,}800$ lb., computed in Section 6. With a column width of 1 ft. 6 in., the moment at the face of the column at B is reduced to

$$91{,}000 - \frac{1}{3} \times 24{,}800 \times 1.5 = 78{,}600 \text{ ft.lb.}$$

The question arises whether the beam to be designed is governed by the moment at the face or by the moment at the centerline of the column. For illustration, refer to Fig. 13 in which a stress distribution is indicated at the faces of the column. The stress T and a considerable part of the stress C will be transmitted across the column by the reinforcement, but

the remaining stresses are distributed over an effective depth which is greater at the centerline than at the face. In European practice, the centerline depth is commonly taken as $d + \frac{a}{6}$, and the stress distribution is assumed to be as indicated in Fig. 13. The relative increase in effective depth at the centerline is then $\frac{a}{6d}$, whereas the relative increase in moment according to Fig. 12 is

$$\frac{\frac{1}{6}Va}{M_C - \frac{1}{3}Va} = \frac{\frac{1}{6} \times \frac{1}{2}wla}{\frac{1}{12}wl^2 - \frac{1}{3} \times \frac{1}{2}wla} = \frac{a}{l - 2a}.$$

The tensile stresses would be greater at the centerline in cases where the increase in moment exceeds the increase in effective depth; that is, when

$$\frac{a}{l - 2a} > \frac{a}{6d} \quad \text{or} \quad d > \frac{l - 2a}{6}.$$

In general, d is smaller than $(l - 2a)/6$; and it is then the moment at the column face that governs the design.

12. Effect of Change in Moment of Inertia

The research made on the effect of moment of inertia, I, of members in continuous frames, is not sufficiently comprehensive, and some discrepancy, therefore, may exist in recommendations for I taken from various sources. In current practice, certain general procedures have been developed which will be briefly discussed.

The ratio of

$$n = \frac{K_C}{K_b} = \frac{I_C \times l}{I_b \times h},$$

in which c refers to columns and b to beams, enters into all analyses of building frames. Considerable uncertainty exists in determining ratios of I_C (for columns) to I_b (for beams).

In typical building frames, the entire cross-sectional area of columns is usually effective in taking stresses, and values of I for columns should preferably be based upon the gross concrete section and computed as $bd^3/12$, in which b denotes width and d depth. It is frequently recommended to add hereto the value of the moment of inertia, I_B, contributed by the longitudinal bar reinforcement, making the total moment of inertia

$$I = \frac{1}{12}bd^3 + \left(\frac{E_s}{E_c} - 1\right) \times I_B.$$

If the column is square and the bars are arranged in a circle, the moment of inertia of the column may be written as:

$$I = \frac{1}{12} \times d^4 + \frac{1}{8}A\left(\frac{E_s}{E_c} - 1\right)c^2, \quad \ldots\ldots \quad (7)$$

in which

d = side dimension of square column,

A = area of longitudinal bars,

c = diameter of circle through centers of bars,

$\frac{E_S}{E_C}$ = ratio of moduli of elasticity of steel and concrete.

The area, A, in equation (7) is assumed uniformly distributed along the circle with diameter c; and by writing the transformation factor for the bars as $\frac{E_S}{E_C} - 1$, it has been taken into account that the area A has been included as concrete in the first term in equation (7).

Cross-sections in beams have both compressive stress and tensile stress. The question arises whether the tensile stresses in the concrete should be ignored. In frame analysis, I is introduced for the purpose of determining slopes and deflections. It is not the particular value of I at any one cross-section, but the I-values over the entire length between joints which govern deflection. Since the uncracked condition generally prevails, the gross area is considered to form the better basis for calculation of I.

The value of I of a T-section is usually taken as $\frac{bd^3}{12}$, in which d is the total depth of the web. The lower limit of b for T-beams is the width of the web and the upper limit is the average value of the panel widths adjacent to the web; but it is improbable that b can be equal to either one of these limiting values. Professor Cross* has suggested that b be taken as the web width, b', times a multiplier, C, thus allowing for the effect of the flange. In other words, the moment of inertia of any T-beam may be written as

$$I = C \times \left(\frac{b'd^3}{12}\right), \quad \ldots \ldots \ldots \ldots \quad (8)$$

in which the quantity in parenthesis equals the moment of inertia of the rectangular web section. For rectangular beams, C therefore equals unity.

Values of C for T-beams, computed on basis of the gross (uncracked) sections, are listed in Table 3. The values of 2, 5, 10 and 20 are ratios of "Flange Width" to "Web Width," and the values of 0.2, 0.3, 0.4 and 0.5 are ratios of "Flange Thickness" to "Web Depth."

Values of C		Flange Width / Web Width: 2	5	10	20
Flange Thickness / Web Depth	0.2	1.3	1.9	2.3	2.7
	0.3	1.4	1.9	2.3	2.7
	0.4	1.4	1.9	2.4	†
	0.5	1.4	2.0	†	†
	Average	1.4	1.9	2.3	2.7

Table 3. Values of C

*See Reference 6.
†These conditions seldom exist in typical designs.

Table 3 covers practically the entire range of flange conditions in typical building frames; and yet, the variation in the value of C is surprisingly small. When the flange width is an uncertain quantity, it is justifiable to consider C a constant, and the value of $C = 2$ is a good average. Inserting $C = 2$ in equation (8) gives the general equation for T-beams:

$$I = \tfrac{1}{6} \times b'd^3, \qquad (9)$$

in which

I = moment of inertia of T-beam
b' = width of web
d = total depth of web

If the ratio of flange width, b, to web width, b', is known, equation (8) is recommended for use with the average values for C given in the last line of Table 3. Equations (8) and (9) may be used directly to obtain values of I, since it is seldom considered justifiable to make any allowance for the reinforcement in beams.

When there is more than one beam in a panel for each column, some designers compute the moment of inertia as the sum of I-values taken according to equations (8) or (9), each term covering one beam web. Not more than one of the parallel beams frames into each column; but—it is contended—the other beams frame into a girder which, in turn, frames into the column. In this way, all parallel beams in the panel must affect the relative stiffness between the column and the floor construction. It seems more reasonable, however, to include a fraction only of the I-values for beams not framing directly into the columns.

If cross-sections are not known when the analysis begins, the designer has to estimate the value of n. It is then important to ascertain how much a discrepancy in n affects the results.

Studies of the effect of varying the values of n are given for certain numerical cases in Section 4. Summaries of the results are given in Figs. 4, 5 and 6, in which moments are plotted as ordinates, and the abscissas represent values of n from $\frac{1}{2}$ to $4\frac{1}{2}$, a range which includes most practical cases. The moments that change the most with varying values of n are those at the end of the beam at the exterior column. They decrease from 71,800 for $n = 4\frac{1}{2}$ to 41,800 for $n = \frac{1}{2}$. Even in this case, a change in n from $4\frac{1}{2}$ to $\frac{1}{2}$ (in the ratio of 9:1) will decrease the moment in the ratio of about 5:3 only.

For all other moments in Figs. 4, 5 and 6, the effect of varying the values of n is surprisingly small. It is seen by inspection that the moments computed in Section 4 and given in the accompanying table are not greatly affected by relatively large variations in the ratio of I_c/I_b.

$n = \frac{I_c}{I_b} \times \frac{l}{h}$		$4\frac{1}{2}$	$\frac{1}{2}$
End Moment	(Interior Column) . . .	77,600 (100)	86,500 (111)
	(First Interior Column)	78,100 (100)	90,100 (115)
Center Moment	(Interior Span)	40,300 (100)	50,000 (124)
	(Exterior Span)	42,500 (100)	51,000 (120)

For the moments in the table, the change with varying value of n ranges from 11 to 24 per cent.

13. *Effect of Distant Loads*

In Fig. 2, only a small part of the entire frame is considered in the analysis; this makes the procedure especially convenient to apply. The fact that the analysis may be applied to part of a frame of such limited extent is in a measure due to the smallness of the effect of distant loads.

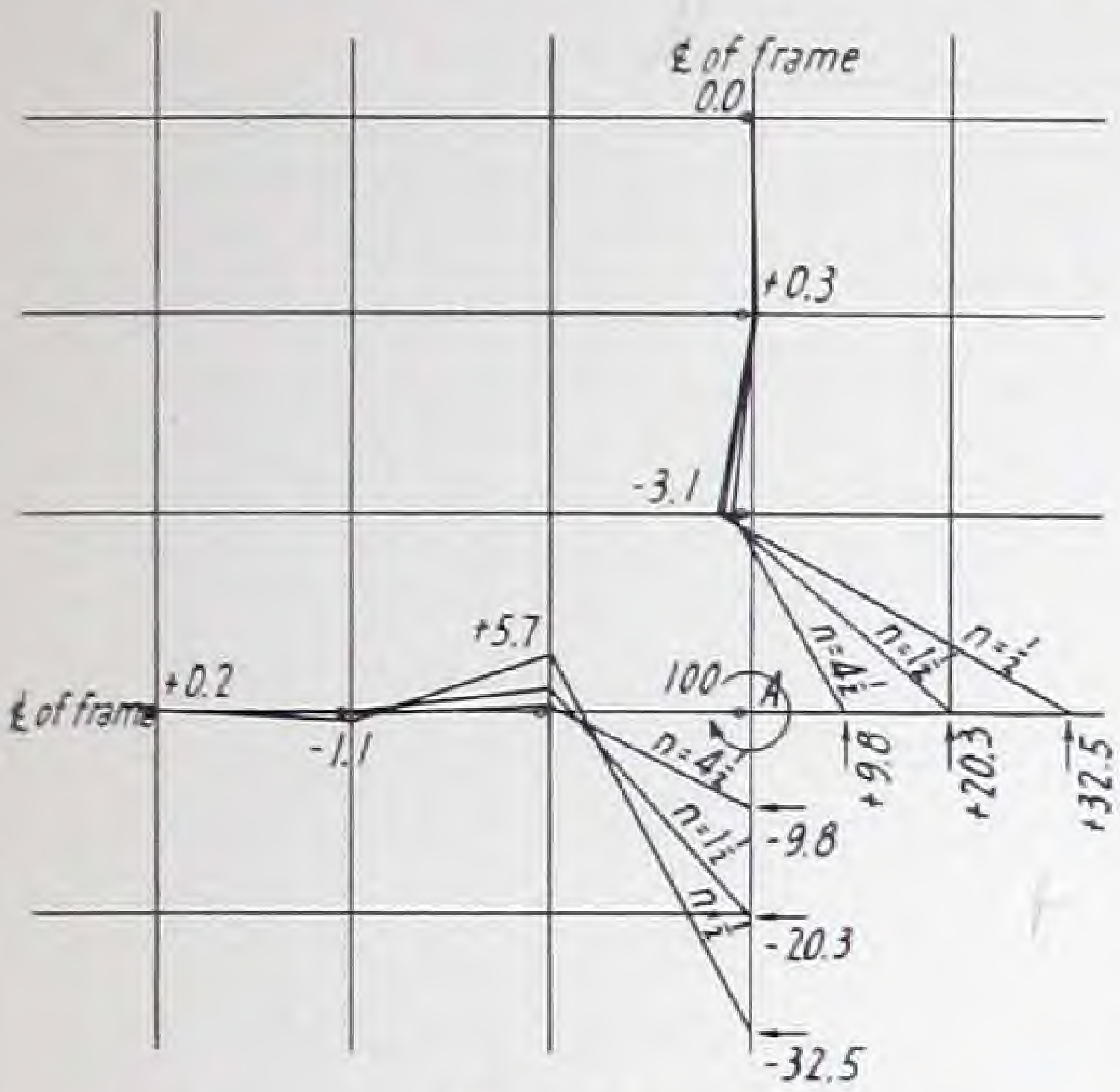


Fig. 14

For illustration, let the vertical and horizontal lines in Fig. 14 represent a building frame in which joint A is rotated by an unbalanced exterior moment of 100. No other load will be considered on the frame; and symmetry is assumed with respect to joint A. The unbalanced moment at A will induce moments in the frame, and the end moments induced at the points immediately to the left of each joint on the horizontal and vertical lines through A are plotted in Fig. 14 for values of $n = \frac{1}{2}$, $1\frac{1}{2}$ and $4\frac{1}{2}$. In the horizontal line through A, the end moments induced for $n = \frac{1}{2}$ are -32.5, $+5.7$, -1.1 and $+0.2$ at distances from A of 0, 1, 2 and 3 times the span length. Adjacent to the column through A, the end moments induced for $n = \frac{1}{2}$ are -32.5, $+3.1$, -0.3 and 0.0 at distances of 0, 1, 2 and 3 times the story height. The induced end moments become smaller when n increases; they approach zero when n approaches infinity (fixed end beams).

At a certain joint in a frame as in Fig. 14, the moments induced by unbalanced moments two or more spans or stories from the joint are small. The rapid rate of decrease of induced moment with increasing distance to point of application of unbalanced moment is a contributing reason why good accuracy may be obtained with a procedure as that in Fig. 2.

14. *Effect of Haunching*

Haunching of beams will affect the fixed end moments produced by the loading and also the stiffness, which for prismatic beams is expressed as I/l. When analyzing frames for horizontal loads such as wind pressure, the stiffness, S, should be corrected for haunching by multiplying it with a factor which may be selected from Fig. 15 in accordance with the shape of the haunches.

For example, let $D/d = 2$ and $a = 0.3$ for both ends of a beam AB as shown in the figure inserted in the left side of Fig. 15. The adjusted value of the stiffness S, taken from the left chart in Fig. 15 for $D/d = 2$ and $a = 0.3$, equals $2.75(I_d/l)$, or the stiffness of this haunched beam is 2.75 times the stiffness of a prismatic beam with moment of inertia $= I_d$ and span length $= l$.

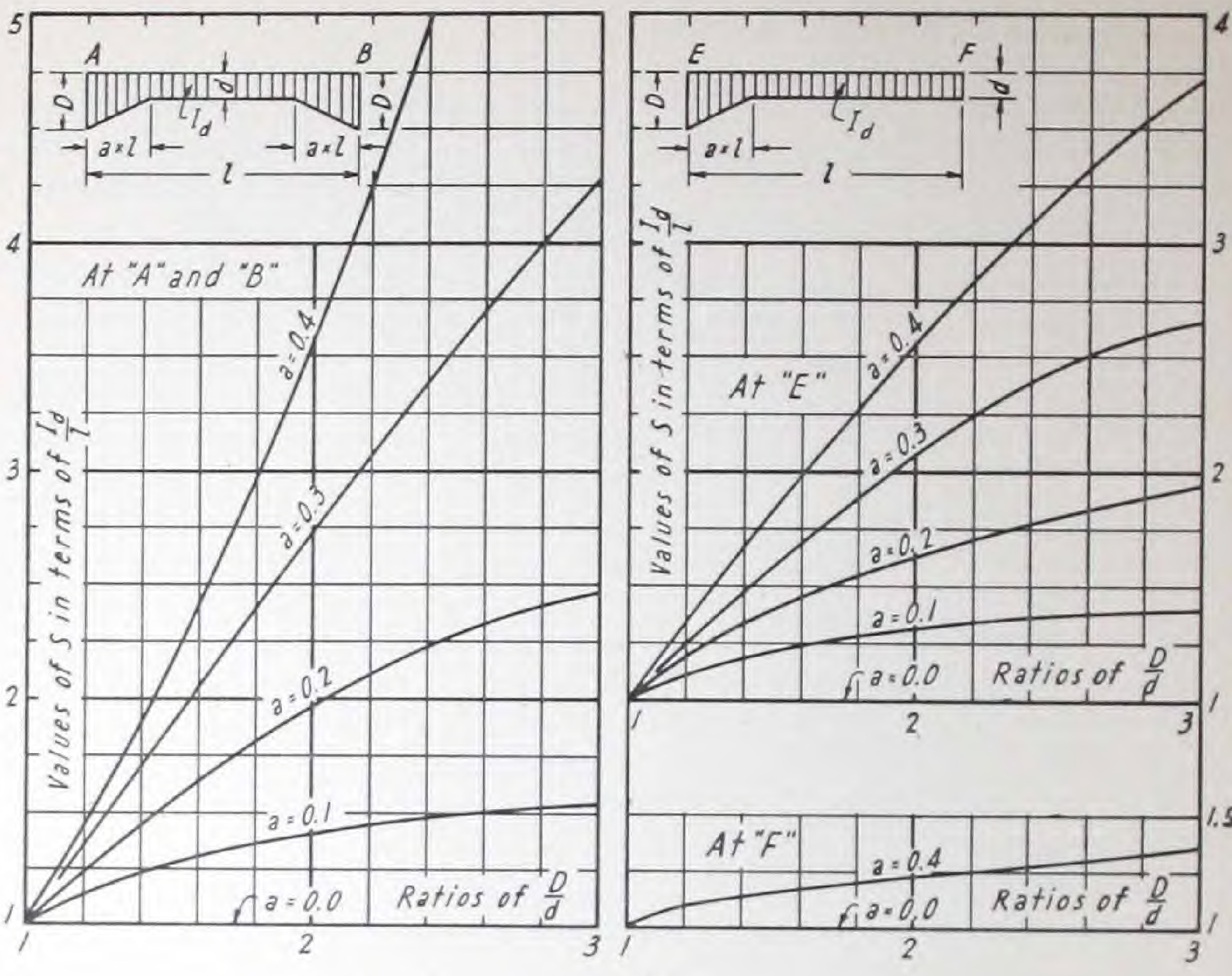


Fig. 15

For vertical loads, the value of I/l need seldom be corrected for haunching in practical design problems; but the fixed end moments should be adjusted to allow for haunching. For the adjustment, Professor Cross has found it surprisingly satisfactory to use the following simple approximation. For a symmetrically haunched beam with dimensions as shown in Fig. 16(a), the fixed end moment may be taken as $(1 + ab)$ times the fixed end moment in a prismatic beam with the same span and loading. When the beam is haunched at one end only, the multiplier may be taken as $(1 + 2ab)$ at the haunched end and $(1 - ab)$ at the prismatic end.* The effects of haunching both ends are additive.

Problem 8. A beam AB with fixed ends has unsymmetrical haunches as shown in Fig. 16(b) and carries a uniformly distributed load, w. Determine the fixed end moments.

Using the products of ab computed in Fig. 16(b) and the multipliers, $(1 + 2ab)$ and $(1 - ab)$, the fixed end moments are adjusted for haunching as follows:

at A:

$$(^1/_{12}wl^2) \times (1 + 2 \times \tfrac{1}{4}) \times (1 - 1 \times 3/40) = 0.116wl^2 = 1.39 \times {}^1/_{12}wl^2,$$

at B:

$$(^1/_{12}wl^2) \times (1 - 1 \times \tfrac{1}{4}) \times (1 + 2 \times 3/40) = 0.072wl^2 = 0.86 \times {}^1/_{12}wl^2.$$

*See Reference 5, in which Table 2 on Page 676 gives numerous comparisons between approximate and exact moments for both straight and parabolic types of haunches.

The moment is 39 per cent greater at A and 14 per cent smaller at B than the moment, $wl^2/12$, for a prismatic beam with the same load and span. The effect of haunching is considerable in this example and should not be ignored. Moments computed by the Column Analogy Method* are

$$0.115wl^2 \text{ at } A.$$

$$0.080wl^2 \text{ at } B.$$

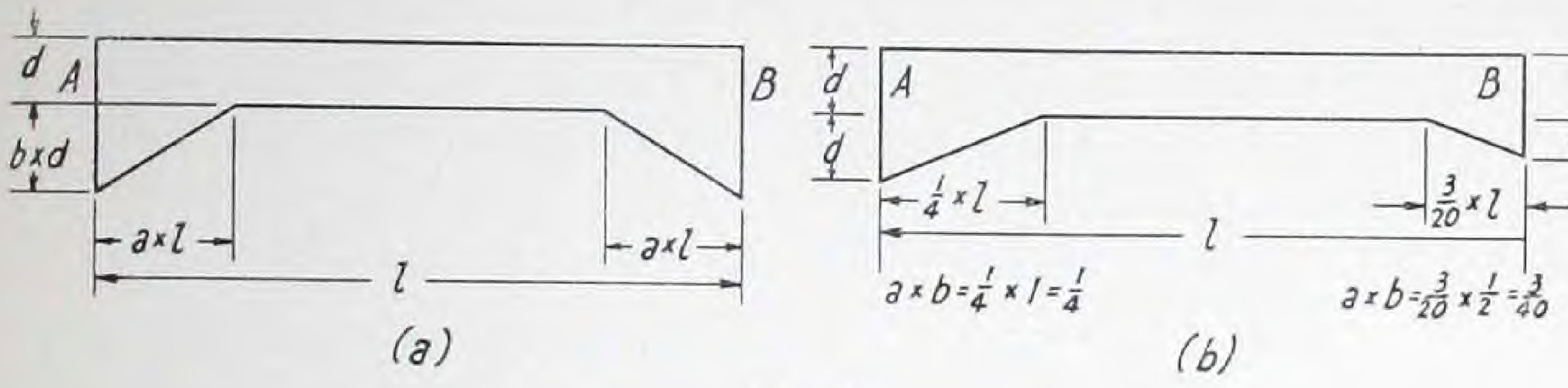


Fig. 16

WIND PRESSURE

15. Introduction

A single vertical frame subject to lateral load may be analyzed with a high degree of accuracy by procedures such as the slope-deflection method. In buildings, however, a number of such vertical frames—bents—act together, and the distribution of load to each frame must be determined before the analysis can proceed. This distribution is as important as the subsequent analysis.

One of the current approximate methods of analysis is based upon the assumptions that the point of contraflexure is at midpoint of each member and that vertical shears in beams are equal.** These assumptions alone give no indication as to the amount of load carried by each bent. The assumption of contraflexure at midpoints becomes very useful, however, when combined with the assumptions presented in Appendix B.

Bents subject to wind pressure have (1) shear deflection which is caused by bending in individual members, and (2) moment deflection, which is due to extension—shortening or lengthening—of column members. Shear deflection is predominant except in tall and narrow, tower-like structures. Moment deflection is ignored in most methods of analysis*** and is also disregarded in the analytical procedure in this text. It should be noted, however, that it is a source of error to ignore column extension, and the error may offset the accuracy of elaborate methods of analysis.

*See Reference 8.

**First put in print in "Steel Construction" by H. J. Burt. Similar assumptions form the basis of R. Fleming's "Portal Method" applying to wind pressure analysis, see page 111, Reference 26.

***Regarding allowance for moment deflection in tall buildings and towers, see Reference 28.

A procedure is presented for determination of shears and moments in buildings subject to lateral loading. The derivation—given in Appendix *B* —is based upon assumptions originally suggested by Professors Wilson and Maney.* The method of distributing wind pressure to the individual bents in this text is similar to one presented by Mr. Albert Smith.** Methods of analyzing building frames for wind pressure will be illustrated and discussed in the numerical examples which follow.

16. Illustrative Problem No. 9

Let Fig. 17 be the framing plan for a floor twenty stories below the roof, the height of each story being 10 ft. Let the direction of the wind be East-West and its intensity 20 p.s.f. All bays are 20 ft. long. The relative values of *I*, moment of inertia, and *K* will be taken as follows:

	I: Relative Value of Moment of Inertia	*K*: Ratio of *I* to Span Length or Story Height
Spandrel beams***	20	20/20 = 1
Interior beams	30	30/20 = 1.5
Wall columns***	40	40/10 = 4
Interior columns	80	80/10 = 8

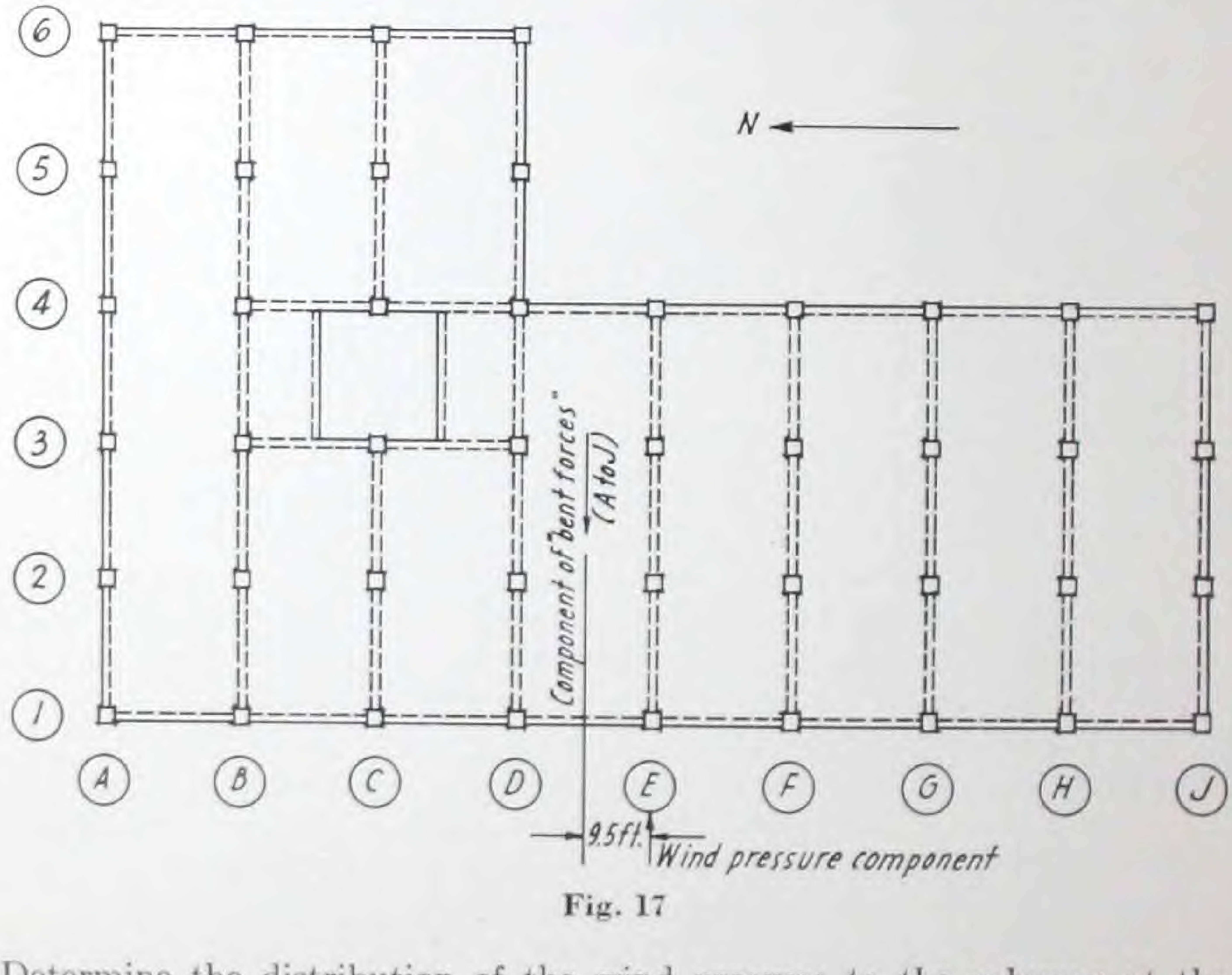


Fig. 17

Determine the distribution of the wind pressure to the columns at the floor shown in Fig. 17.

*See page 24 in Reference 21.
**See References 33 and 36.
***An adjustment will be made later in the values of *I* and *K* for bent *J*.

The nine frames from A to J in the East-West direction will assist in resisting the total wind pressure above the floor, which, with 8 bays 20 ft. long and 20 stories 10 ft. high equals

$$W = (8 \times 20) \times (20 \times 10) \times 20 = 640{,}000 \text{ lb.}$$

Each column in Fig. 17 is part of a frame extending East-West and will carry wind shear. The proportion of the total wind pressure taken by each column is proportional to a "joint coefficient" which is a function of the K-values of members in the bent. The K-values must be known or selected before the analysis can proceed. According to the assumption and the derivation in Appendix B, the coefficient at any joint may be determined as follows:

$$\text{Joint Coefficient} = (K \text{ for col.}) \times \frac{\text{Sum of } K\text{'s for adjacent beams}}{\text{Sum of } K\text{'s for adjacent members}}. \quad (10)$$

For the floor in Fig. 17 and the K-values given above, joint coefficients computed by equation (10) are recorded in the second column in Table 4. At joint $C3$, for example, the K-value is 8 for the columns, 1.5 for the beam between $C2$ and $C3$, and zero between $C3$ and $C4$. The coefficient at $C3$ equals

$$8 \times \frac{1.5 + 0}{1.5 + 0 + 8 + 8} = 0.69\,.$$

The next step is to compute the sum of the coefficients for each bent at the floor, a sum that represents the force with which the bent resists a unit translation. The center of gravity of these "bent forces" is then computed by multiplying each force by its distance from bent J and dividing the sum of these products by the sum of the forces. The distance from bent J to the center of gravity equals $3458/38.61 = 89.5$ ft., the values in parentheses being used for bent J (see Table 4).

The distance from bent J is 89.5 ft. to the center of gravity of the bent forces, but 80.0 ft. to the center of gravity of the wind pressure. An eccentricity of 9.5 ft. exists which tends to twist the entire building frame with respect to some vertical axis. The twisting or torsional moment may be eliminated by determining, by trial, new K-values for bent J of such magnitude that the eccentricity becomes negligible. Bent J is chosen in this case because it is farthest from the center of gravity, thereby giving the change in J great weight; and bent J may be assumed to be a wall in which architectural design may readily be adjusted to suit structural demands. In Table 4, K-values have been trebled for all members in bent J*. By this change, the distance to the center of gravity is reduced from 89.5 ft. to $3458/43.59 = 79.4$ ft., and the eccentricity reduced from 9.5 ft. to 0.6 ft.

By adjusting the J-bent and making the eccentricity negligible, the advantage is gained that all joints in the floor get the same translation. This means that the column shears, V_A, are proportional to the joint coeffi-

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*By stiffening bent J, the beam shear in J (see Table 4) is made twice as large as that in the adjacent bents. The uplift on the windward column, $J1$, is therefore twice that on the windward columns in adjacent bents. It is important to ascertain that the uplift in column $J1$ is not too large compared with the dead load available to counteract it. In case the uplift is too large, the stiffness of members must be adequately decreased in bent J and increased commensurately in the adjacent bents.

cients in the second column of Table 4, the proportionality factor being equal to

$$\frac{640{,}000}{43.59} = 14{,}700,$$

in which the numerator is the total wind pressure and the denominator the sum of all joint coefficients.

Joint	Joint Coefficient	Columns Shear lb.	Columns Moment ft.lb.	Beams Moment ft.lb.	Beams Shear lb.
A1	$4\times\frac{0+1}{0+1+4+4}=.44$	6,500	32,500	65,000	6,200
A2	$4\times\frac{1+1}{1+1+4+4}=.80$	11,800	59,000	59,000	5,900
A3	A2 .80	same	as joint	A2	5,900
A4	A2 .80	same	as joint	A2	5,900
A5	A2 .80	same	as joint	A2	6,200
A6	A1 .44	same	as joint	A1	
	4.08	4.08×8×20=653			
B1	$4\times\frac{0+1.5}{0+1.5+4+4}=.63$	9,300	46,500	93,000	9,280
B2	$8\times\frac{1.5+1.5}{1.5+1.5+8+8}=1.26$	18,500	92,500	92,500	9,250
B3	B2 1.26	same	as joint	B2	9,250
B4	B2 1.26	same	as joint	B2	9,250
B5	B2 1.26	same	as joint	B2	9,280
B6	B1 .63	same	as joint	B1	
	6.30	6.30×7×20=882			
C1	B1 .63	same	as joint	B1	9,280
C2	B2 1.26	same	as joint	B2	9,680
C3	$8\times\frac{1.5+0}{1.5+0+8+8}=.69$	10,100	50,500	101,000 / 0	0
C4	C3 .69	10,100	50,500	0 / 101,000	9,680
C5	B2 1.26	same	as joint	B2	9,280
C6	B1 .63	same	as joint	B1	
	5.16	5.16×6×20=620			
D1	B1 .63	same	as joint	B1	9,280
D2	B2 1.26	same	as joint	B2	9,250
D3	B2 1.26	same	as joint	B2	9,380
D4	$8\times\frac{1.5+1}{1.5+1+8+8}=1.08$	15,900	79,500	95,000 / 64,000	6,150
D5	A2 .80	same	as joint	A2	6,200
D6	A1 .44	same	as joint	A1	
	5.47	5.47×5×20=547			

Joint	Joint Coefficient	Columns Shear lb.	Columns Moment ft.lb.	Beams Moment ft.lb.	Beams Shear lb.
E1	B1 .63	same	as joint	B1	9,280
E2	B2 1.26	same	as joint	B2	9,250
E3	B2 1.26	same	as joint	B2	9,280
E4	B1 .63	same	as joint	B1	
	3.78	3.78×4×20=302			
F1	B1 .63	same	as joint	B1	9,280
F2	B2 1.26	same	as joint	B2	9,250
F3	B2 1.26	same	as joint	B2	9,280
F4	B1 .63	same	as joint	B1	
	3.78	3.78×3×20=227			
G1	B1 .63	same	as joint	B1	9,280
G2	B2 1.26	same	as joint	B2	9,250
G3	B2 1.26	same	as joint	B2	9,280
G4	B1 .63	same	as joint	B1	
	3.78	3.78×2×20=151			
H1	B1 .63	same	as joint	B1	9,280
H2	B2 1.26	same	as joint	B2	9,250
H3	B2 1.26	same	as joint	B2	9,280
H4	B1 .63	same	as joint	B1	
	3.78	3.78×1×20=76			
J1	(A1:0.44) 1.33	19,500	97,500	195,000	18,580
J2	(A2:0.80) 2.40	35,300	176,500	176,500	17,650
J3	(A2:0.80) 2.40	same	as joint	J2	18,580
J4	(A1:0.44) 1.33	same	as joint	J1	
	(2.48) 7.46	7.46×0×20=00			

Sum of all joint coefficients: 43.59
Moment of joint coefficients with respect to bent J: 3458
Eccentricity of joint coefficients: $80.0-\frac{3458}{43.59}=0.6$ ft
Factor for joint coefficients: $\frac{640{,}000}{43.59}=14{,}700$

Table 4. Distribution of Wind Pressure to Frames Extending East-West

Multiplying each joint coefficient by 14,700 gives the column shears recorded in the third column. Column moments are computed as the column shear times $h/2$, or 5 ft. At each joint, the sum of the column moments equals the sum of the beam moments and will be distributed to the beams in proportion to their K-values. Beam shear is computed as the sum of the end moments in the beam divided by the bay length, 20 ft.

Attention is called to joint $D4$, where the sum of the column moments, $2 \times 79{,}500 = 159{,}000$ ft.lb., is distributed to the adjacent beams in proportion to their K-values as follows:

$$\frac{1.5}{1.5 + 1.0} \times 159{,}000 = 95{,}000 \text{ ft.lb. to the interior beam, and}$$

$$\frac{1.0}{1.5 + 1.0} \times 159{,}000 = 64{,}000 \text{ ft.lb. to the spandrel beam.}$$

Table 5 contains a summary of a study made by combining the floor framing in Fig. 17 with columns of varying stiffness. The original K-values, 4 for wall columns and 8 for interior columns, were reduced to 2 and 4 and also to 1 and 2, the floor framing remaining unchanged. For these three designs, the percentage of wind pressure carried by the individual bents was computed, the translation at all joints in a floor being equal. The results are recorded in Table 5.

Bent	K-values of Columns		
	1 and 2	2 and 4	4 and 8
A	9.0	9.2	9.4
B	14.4	14.4	14.4
C	12.4	12.0	11.8
D	12.6	12.6	12.5
E	8.7	8.7	8.7
F	8.7	8.7	8.7
G	8.7	8.7	8.7
H	8.7	8.7	8.7
J	16.8	17.0	17.1

Table 5

The percentage of wind pressure carried by each bent is surprisingly uniform despite the variation in the K-values of the columns. This uniformity will greatly reduce the analytical work required for a group of typical floors.

17. Eccentric Wind Pressure

If the original K-values for bent J as given in parentheses in Table 4 are maintained, the floor in addition to its translation due to wind pressure will be rotated by a torsional moment equal to the wind pressure multiplied by the eccentricity of 9.5 ft. (see Fig. 17), or

$$M = 640{,}000 \times 9.5 = 6{,}080{,}000 \text{ ft.lb.}$$

The columns resist this torsion and the column shears computed in Table 4 should be corrected accordingly.

Mr. Albert Smith, in his paper on "Wind Bracing,"* uses the following method of correction. He lets the torsional moment be resisted by shear at all the joints of all bents in *both* directions and determines the shear correction, V_X, in each bent, X, in accordance with the formula

$$V_X = \frac{M}{I_X + I_Y} v_X x, \quad \ldots \ldots \ldots \ldots \quad (11)$$

in which

X = Bents A to J,
v_X = Relative shear in bents X (recorded in Tables 4 and 6),
x = Distance from bent X to center of gravity of v_X (recorded in Table 6),
$I_X = v_X x^2$
Y, v_Y, y and I_Y = similar values for Bents 1, 3, 4, and 6.

Equation (11) will, for illustration, be applied to the condition in Problem 9, using the original K-values in bent J.

Refer to Problem 9 for values of v_X and the position of the center of gravity. The calculations leading to v_Y and y are omitted, but may be duplicated from data in Problem 9. The values of I_X and I_Y in equation (11) are computed in Table 6.**

Bent	$v_X \times x^2 = I_X$	Bent	$v_Y \times y^2 = I_Y$
A	$4.08 \times 70.5^2 =$ 20,300	1	$6.48 \times 41.3^2 =$ 11,100
B	$6.30 \times 50.5^2 =$ 16,100	3	$2.64 \times 1.3^2 =$ 00
C	$5.16 \times 30.5^2 =$ 4,800	4	$6.67 \times 18.7^2 =$ 2,300
D	$5.47 \times 10.5^2 =$ 600	6	$2.48 \times 58.7^2 =$ 8,500
E	$3.78 \times 9.5^2 =$ 300		
F	$3.78 \times 29.5^2 =$ 3,300		
G	$3.78 \times 49.5^2 =$ 9,300		$I_Y =$ 21,900
H	$3.78 \times 69.5^2 =$ 18,200		$I_X =$ 92,800
J	$2.48 \times 89.5^2 =$ 19,900		$I_X + I_Y =$ 114,700
	$I_X =$ 92,800		

Table 6

After having determined $I_X + I_Y$, the regular procedure would be to compute V_X from equation (11) and then correct the shear in each column; but the following short cut is more convenient. Compute for each bent, see Table 7, the value of

$$F = \frac{640{,}000}{38.61} + \frac{640{,}000 \times 9.5}{114{,}700}(x) = 16{,}600 + 53.0(x),$$

in which 38.61 is the sum of all the joint coefficients in Table 4, using 2.48 for the bent J. The products of F and the joint coefficients in Table 4 equal the final shears including correction for eccentricity.

*See Reference 33.

**Bents 1, 3, 4 and 6 only are included on the right side of Table 6. It is assumed, then, that the floor joists extending North-South in Fig. 17 have no stiffness in resisting wind pressure. In determining stresses due to wind blowing in the North-South direction, some designers make allowance for floor joists and calculate their stiffness by equations (8) or (9) as discussed in Section 12 for cases in which there is more than one beam in a panel for each column.

Bent	$\frac{M}{I_X+I_Y}(x)$	$F=16{,}600 + \frac{M}{I_X+I_Y}(x)$	$\frac{F}{14{,}700}$
A	$53.0\times(-70.5)=-3{,}700$	12,900	0.88
B	$53.0\times(-50.5)=-2{,}700$	13,900	0.95
C	$53.0\times(-30.5)=-1{,}600$	15,000	1.02
D	$53.0\times(-10.5)=-\ 600$	16,000	1.09
E	$53.0\times(+\ 9.5)=+\ 500$	17,100	1.16
F	$53.0\times(+29.5)=+1{,}600$	18,200	1.24
G	$53.0\times(+49.5)=+2{,}600$	19,200	1.31
H	$53.0\times(+69.5)=+3{,}700$	20,300	1.38
J	$53.0\times(+89.5)=+4{,}700$	21,300	—

Table 7

The last column in Table 7 gives ratios of $F/14{,}700$, which is the ratio of column shears with and without eccentricity, see also Table 4. When the eccentricity is 9.5 ft., the shear varies considerably from bent to bent, the shear in bent H, for example, being 38 per cent greater than it is when no eccentricity exists. It is seen that changing the stiffness of a bent may greatly affect the distribution of wind pressure; and for this reason, values of K should include allowance for effect of haunching as discussed in Section 14.

18. Accuracy of Approximate Analysis

In order to study the accuracy of the results obtained by an analysis as applied in Section 16, two frames were analyzed for wind pressure. The first frame is one which Professors Wilson and Maney have analyzed by the slope-deflection method;* the second frame has been analyzed by Mr. John E. Goldberg** by a method of converging approximations. The results of the analyses are given in Fig. 18 for the third floor only; they are typical for most of the floors in the bent.

The moment values in parentheses in Fig. 18 are taken from the sources mentioned. Moments obtained by the procedure used in this text are underscored, and typical computations are as follows.

The joint coefficients in Fig. 18(a) are

$$\text{at exterior columns: } 35.4 \times \frac{21.4}{21.4 + 35.4 + 35.6} = 8.20$$

$$\text{at interior columns: } 35.5 \times \frac{21.4 + 29.2}{21.4 + 29.2 + 35.5 + 35.6} = 14.75$$

Sum of coefficients for four columns: 45.90

The column moments, in in.lb., are obtained by multiplying the joint coefficient by

$$\frac{6{,}690 \times (7 \times 12)}{45.90} = 12{,}240 \text{ for columns above 3rd floor}$$

$$\frac{7{,}140 \times (8 \times 12)}{45.90} = 14{,}930 \text{ for columns below 3rd floor,}$$

in which the values of 6,690 and 7,140 are the wind shears given in Fig. 18.

*See Reference 21.
**See Reference 34.

The moments in beams are obtained by distribution of the sum of the column moments; thus, the end moment in the center span is determined as

$$(181{,}000 + 220{,}000) \times \frac{29.2}{29.2 + 21.4} = 231{,}000 \text{ in.lb.}$$

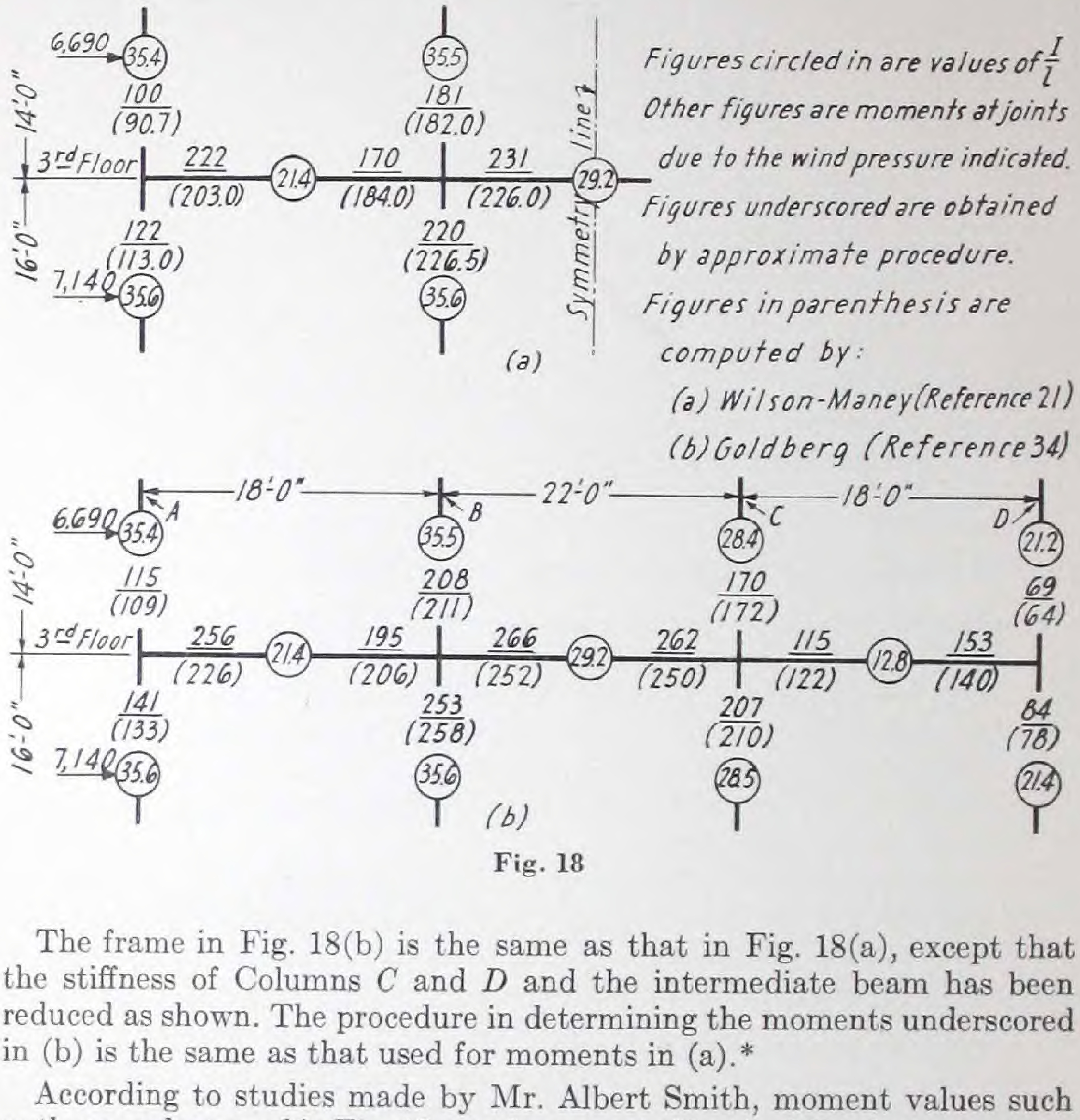

Fig. 18

The frame in Fig. 18(b) is the same as that in Fig. 18(a), except that the stiffness of Columns *C* and *D* and the intermediate beam has been reduced as shown. The procedure in determining the moments underscored in (b) is the same as that used for moments in (a).*

According to studies made by Mr. Albert Smith, moment values such as those underscored in Fig. 18 may be corrected by the following procedure: (1) compute a correction equal to one-fourth of the difference between the larger and the smaller moment at the ends of a member (beam or column), (2) deduct the correction from the larger moment, and, (3) add the correction to the smaller moment. Consider, for illustration, the end span in Fig. 18 (a) and compute: (1) the correction $\frac{1}{4}$ (222 − 170) = 13, (2) the corrected moment to the left 222 − 13 = 209, (3) the corrected moment to the right 170 + 13 = 183. By such corrections, some allowance may be made for the fact that the point of contraflexure is not at the midpoint as assumed in the derivation.

*The moments in parentheses in Fig. 18(b) do not satisfy the requirement that their sum equals zero at each joint. At column *A*, for instance, the sum of column moments, 109 + 133 = 242, should equal the beam moment, but the beam moment is given as 226. These moment values are approximate.

19. *Moment Deflection and the Ideal Layout*

The beam shears in Table 4 are practically equal in all spans of a regular bent. In such bents, the sum of shears is small at the interior columns, and these columns will receive little or no direct load from wind pressure. The direct load in the exterior columns equals the shear in the end span and is about 9,300 lb. in bents, *B*, *C* and *E* to *H*, the direct load being tension in the windward and compression in the leeward column. The result is a differential column extension and a warping of the floor, which causes a "secondary" distribution of shears and moments.

It is possible to arrange a structural layout so as to minimize the effect of differential column extension and thus approach an "ideal layout" which may be accurately analyzed. In the ideal layout there is (1) no wind pressure eccentricity and (2) no warping of the floors. The building frame should be laid out with symmetry in both directions; or, at least, the wind pressure component should go through the center of gravity of the joint coefficients, a case which was illustrated in Problem 9. To avoid warping the floors, the interior bays should be designed to carry much larger shears than the end bays. This may be accomplished by making the coefficients at exterior joints sufficiently small compared with the coefficients at interior joints.

To illustrate the procedure in minimizing the warping of floors, *K*-values for bent *B* in Fig. 17 and Table 4 will be adjusted as follows and the wind pressure distributed accordingly, the calculations being recorded in Table 8.

Span	*B*1-*B*2, *B*5-*B*6	*B*2-*B*3, *B*4-*B*5	*B*3-*B*4
K-value	1.0	1.5	1.7

Bent *B* is assumed to receive the same total wind pressure, 92,600 lb., as in Table 4; and shears, moments and direct column loads are computed in Table 8 by the same procedure as used in Table 4.

Joint	Joint Coefficient	Columns		Beams		Direct Column Load	
		Shear	Moment	Moment	Shear	Adjusted	Original
*B*1	$4\times\frac{0+1}{0+1+4+4}=0.444$	7,200	36,000	72,000		+7,100	+9,300
					7,100		
*B*2	$8\times\frac{1+1.5}{1+1.5+8+8}=1.080$	17,500	87,500	70,000 105,000		+3,220	0
					10,320		
*B*3	$8\times\frac{1.5+1.7}{1.5+1.7+8+8}=1.333$	21,600	108,000	101,300 114,700		+1,150	0
					11,470		
*B*4	1.333	21,600	108,000	114,700 101,300		−1,150	0
					10,320		
*B*5	1.080	17,500	87,500	105,000 70,000		−3,220	0
					7,100		
*B*6	0.444	7,200	36,000	72,000		−7,100	−9,300

Table 8

The direct column loads determined for the original and also for the adjusted K-values (see Table 8) are plotted in Fig. 19. The curve marked "ideal" in Fig. 19 indicates the magnitude of direct column loads that will cause no warping of the floor and, therefore, no secondary distribution of moments and shears. It is seen from the curves in Fig. 19 that a much better condition for accurate wind pressure analysis is created by selecting proportions of beams in accordance with the K-values suggested. From inspection of the joint coefficients in Table 8, it is seen that the adoption of new column sections may be similarly effective. It is especially advisable to make the wall columns, $B1$ and $B6$, rectangular with the smaller dimension in the East-West direction.

The effect of warping of floors becomes increasingly serious the shorter the outer span is compared with interior spans. In such cases, warping may be minimized and the shear reduced in end beams by judicious selection of shallow outer beams and narrow exterior columns.

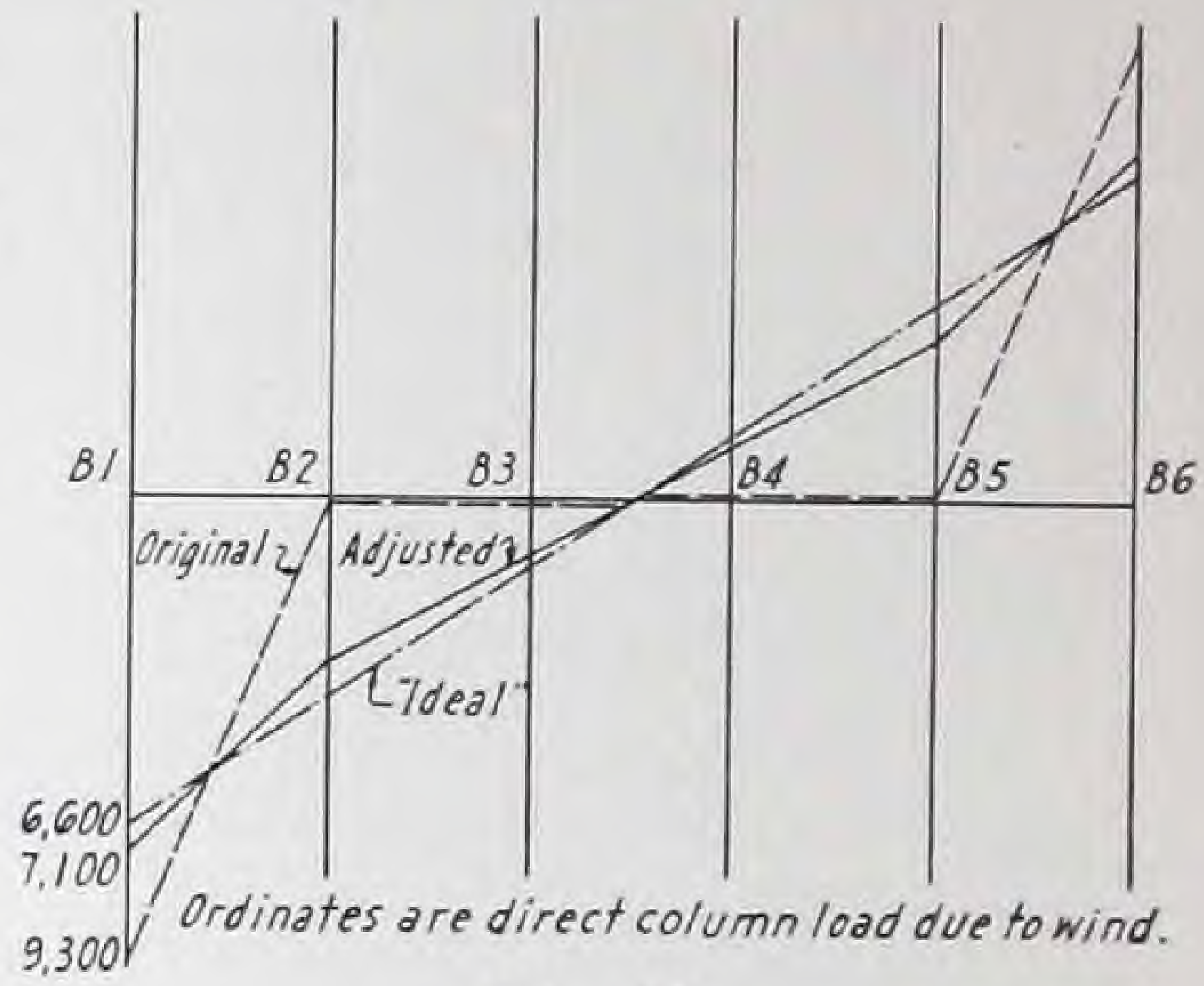

Fig. 19

20. *Illustrative Problem No. 10; Vertical Load and Wind Pressure Combined*

The problem is in two parts: (A) Analysis of bent G in Fig. 17 for vertical loads, and (B) determination of maximum moments and shears due to vertical loads and wind pressure combined.

(*A*) *Analysis for Vertical Loads.* Bent G in Fig. 17 has three spans, each of which has a length of 20 ft. from center to center of the columns; the width of the bay carried by bent G is 20 ft. The successive steps in the analysis will be the determination of

(a) Loads.
(b) Values of K or n.
(c) Values of joint coefficients and moment coefficients.
(d) Fixed end moments.
(e) Moments at centerlines of columns.
(f) Moment reduction for effect of width of columns.
(g) Shear.
(h) Maximum values required for design.

(a) The *loads* in lb. per lin. ft. of the beams will be taken as

$$\begin{aligned} \text{Dead Load (D.L.)} &= 75 \times 20 + 300 = 1800 \\ \text{Live Load (L.L.)} &= 100 \times 20 = 2000 \\ \text{Total Load (T.L.)} &= 3800 \end{aligned}$$

(b) The K-values* will be the same as those used in the wind analysis of Bent G (see Problem 9, Section 16):

Wall Columns: $K = 4$
Interior Columns: $K = 8$
Interior Beams: $K = 1.5$

(c) *Values of joint coefficients and moment coefficients* based upon the K-values in (b) are recorded below, see summary in Section 10 for equations and references:

Joint Mark See Fig. 20	$\frac{K_c}{K_b}=n$	Beam Moments		Column Moments	
		Value of Q	At Ext. Col.	Exterior	Interior
		$\frac{1}{4n+2}$	$\frac{4n}{4n+1}$	$\frac{2n}{4n+1}$	$\frac{3n}{6n+2}$
$G1$	$\frac{4}{1.5}$	0.079	0.92	0.46	....
$G2$	$\frac{8}{1.5}$	0.043	....	....	0.47

(d) *Fixed end moments*, in ft.lb., using the coefficient in case 5 in Fig. 1, are**

D.L.: $-\frac{1}{12} \times 1800 \times 20^2 = -60{,}000$
L.L.: $-\frac{1}{12} \times 2000 \times 20^2 = -66{,}700$
T.L.: $-\frac{1}{12} \times 3800 \times 20^2 = -126{,}700$

(e) *Moments at centerlines of columns*, computed according to the procedures in Sections 2 and 7, are numbered below. Corresponding numbers are written on Fig. 20 to indicate the point where the moment is taken and the loading arrangement which produces it.

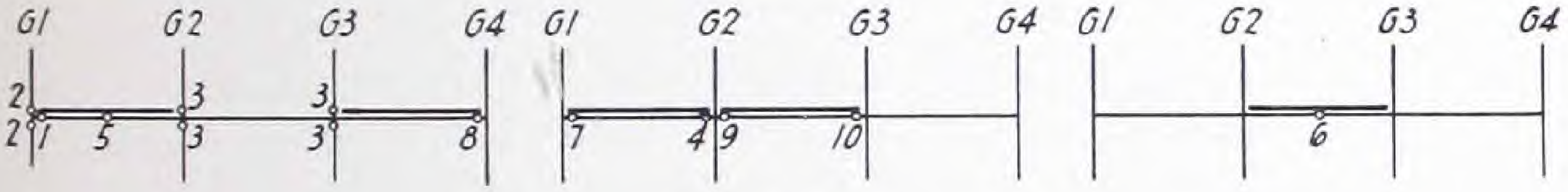

Fig. 20

1. Maximum end moment in $G1-G2$ at $G1$:
$M_1 = 0.92 \times (-126{,}700) = -116{,}500$

2. Maximum column moment at $G1$:
$M_2 = 0.46 \times 126{,}700 = \underline{58{,}300}$

3. Maximum column moment at $G2$:
$M_3 = 0.47 \times 66{,}700 = \underline{\underline{32{,}000}}$

4. Maximum end moment in $G1-G2$ at $G2$:
$M_4 = -126{,}700 - 1 \times 0.079 \times 126{,}700 = 136{,}800$

*When cross-sections are not known, select values for Q in accordance with the classification suggested in Section 3. If the cross-sections are known, values of I—and K—may be determined by using equations (7), (8) or (9) in Section 12.
**For numerical example involving unequal spans, see Problem 7.

5. Maximum center moment in $G1-G2$:*

$$
\begin{aligned}
\text{at } G1&: -126{,}700+2\times0.079\times126{,}700+1\times0.043\times(-66{,}700) = -109{,}500\\
\text{at } G2&: -\underline{126{,}700-1\times0.079\times126{,}700-2\times0.043\times(-66{,}700) = -131{,}100}\\
\text{Average}&: \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -120{,}300
\end{aligned}
$$

$$M_5 = +\tfrac{1}{8}\times 3800 \times 20^2 - 120{,}300 = +69{,}700$$

6. Maximum center moment in $G2-G3$:**

at $G2$ and $G3$:

$$-126{,}700 + (2\times 0.079 - 1\times 0.079)\times 66{,}700 = -121{,}400$$

$$M_6 = +\tfrac{1}{8}\times 3800\times 20^2 - 121{,}400 = +68{,}600$$

7. End moment at $G1$ to be used for maximum shear at $G2$:

$$M_7 = -126{,}700 + 2\times 0.079\times 126{,}700 = -106{,}600$$

(f) *Moments are reduced* for effect of width of columns as follows, the column dimension being taken as 2 ft. 6 in. in the direction from $G1$ to $G4$ ($a = 2.5$, $V = 3800 \times 10 = 38{,}000$, see Section 11):

$$
\begin{aligned}
M_1 &= -116{,}500 + \tfrac{1}{3}\times 38{,}000\times 2.5 = -\underline{\ 84{,}800}\\
M_4 &= -136{,}800 + \tfrac{1}{3}\times 38{,}000\times 2.5 = -\underline{105{,}100}\\
M_5 &= +\ 69{,}700 - \tfrac{1}{9}\times 38{,}000\times 2.5 = +\underline{\ 59{,}100}\\
M_6 &= +\ 68{,}600 - \tfrac{1}{9}\times 38{,}000\times 2.5 = +\underline{\ 58{,}000}
\end{aligned}
$$

(g) *Shear* in beams at the face of the columns, the width of which is taken as 2 ft. 6 in., is computed as follows,*** the numbers below referring to points on Fig. 20 with loading arranged to give maximum shear in pounds.

4. Maximum shear in $G1-G2$ at $G2$:

$$V_4 = 3800\times\frac{17.5}{2} + \frac{136{,}800 - 106{,}600}{17.5} = \underline{35{,}000}$$

8. Maximum shear in $G1-G2$ at $G1$ (same as at $G4$):

$$V_8 = 3800\times\frac{17.5}{2} + \frac{109{,}500 - 131{,}100}{17.5} = \underline{32{,}100}$$

9. Maximum shear in $G2-G3$ at $G2$ and $G3$:

$$
\begin{aligned}
M_9 &= -126{,}700 - 1\times 0.043\times 66{,}700\\
\underline{M_{10}} &\underline{= -126{,}700 + 2\times 0.043\times 66{,}700}\\
M_9 - M_{10} &= -3\times 0.043\times 66{,}700 = -8{,}600
\end{aligned}
$$

$$V_9 = 3800\times\frac{17.5}{2} + \frac{8{,}600}{17.5} = \underline{33{,}800}$$

---

*Using the moment coefficient for center moments given in the footnote on page 14 will simplify the calculations as follows: $n = \tfrac{1}{2}\left(\frac{4}{1.5}+\frac{8}{1.5}\right) = 4$; $\frac{n}{21n-2} = \frac{2}{41}$; $M_5 = \frac{2}{41}\times 3800\times 20^2 = 74{,}100$

**Using the moment coefficient for center moments given in the footnote on page 14 will simplify the calculations as follows: $n = \frac{8}{1.5}$; $\frac{n}{21n-2} = \frac{8}{165}$; $M_6 = \frac{8}{165}\times 3800\times 20^2 = 73{,}700$.

***See Section 6: Shear in Beams.

(h) *Maximum* values required for design* are as follows:

Moment in beams: M_1, M_4, M_5, M_6.

Shear in beams: V_4, V_8, V_9.

Direct compression in beams is disregarded.

Moment in columns: M_2, M_3.

Shear in columns is disregarded.

Direct compression in columns taken from the customary computations of column loads needs adjustment because the frame action transfers some column load from $G1$ to $G2$. An increase in column load of 5 per cent (see $V_4 = 33{,}300 + 1{,}700$) at $G2$ is conservative, but no decrease in column load at $G1$ is warranted in this example.

(*B*) *Maximum moments and shears due to vertical load and wind pressure combined.* The frame is the same as that analyzed in (A) for vertical load and in Problem 9, Section 16, for wind pressure.

Using the centerline moments determined in Table 4, the beam moment in ft.lb. at the face of all the columns in Bent G is

$$M_w = \pm \begin{pmatrix} 92{,}500 \\ 92{,}800 \end{pmatrix} \times \frac{20.0 - 2.5}{20.0} = \pm 81{,}000$$

The maximum moments, M_1, M_4 and M_5, in the end span computed in (A) are plotted in Figs. 21(a) and (b) and connected by a parabola, which will be considered the curve of maximum negative and maximum positive moments due to vertical loading.** Beam moments due to wind pressure are indicated by the lines connecting $\pm 81{,}000$ with $\mp 81{,}000$ in Fig. 21(a) and (b). The combined moments are measured between the inclined line and the parabola.

The maximum positive and maximum negative moments taken from Figs. 21(a) and (b) are plotted in Fig. 21(c). It is seen that wind pressure added to vertical load increases the positive moments and also the negative moments at both ends of beam $G1$-$G2$; the length in which there may be tension in the bottom or tension in the top of the beam is also increased.

The maximum shears, V_4 and V_8, are plotted at $G1$ and $G2$ in Fig. 22(a) and (b) and connected with a straight dotted line which will be considered the curve of maximum shear due to vertical loading.**

The shear in the beams at the face of the columns due to wind pressure is constant and equals

$$V_w = \frac{81{,}000 + 81{,}000}{17.5} = 9{,}300$$

The solid, inclined lines in Figs. 22(a) and (b), drawn at a constant distance of 9,300 from the dotted lines, represent the combined shear. The maximum shears taken from Figs. 22(a) and (b) are plotted in Fig. 22(c). It is seen that the shear may be increased over the entire length of the clear span.

*For discussion of *minimum* values of moments and their significance in detailing bar reinforcement in frames with unequal spans, see page 22 in Section 9.

**In the intervals between the points plotted, moments and shears may be numerically larger than those indicated. The discrepancies are frequently small and will be disregarded in this case.

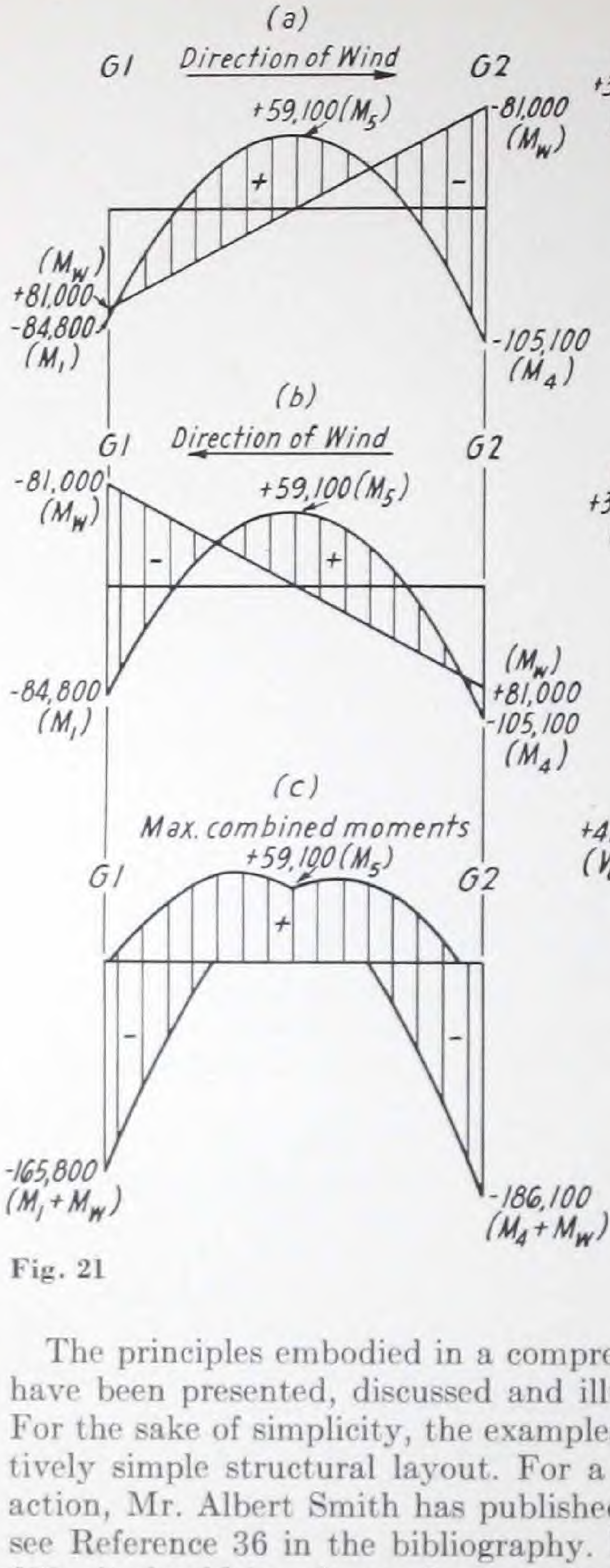

Fig. 21

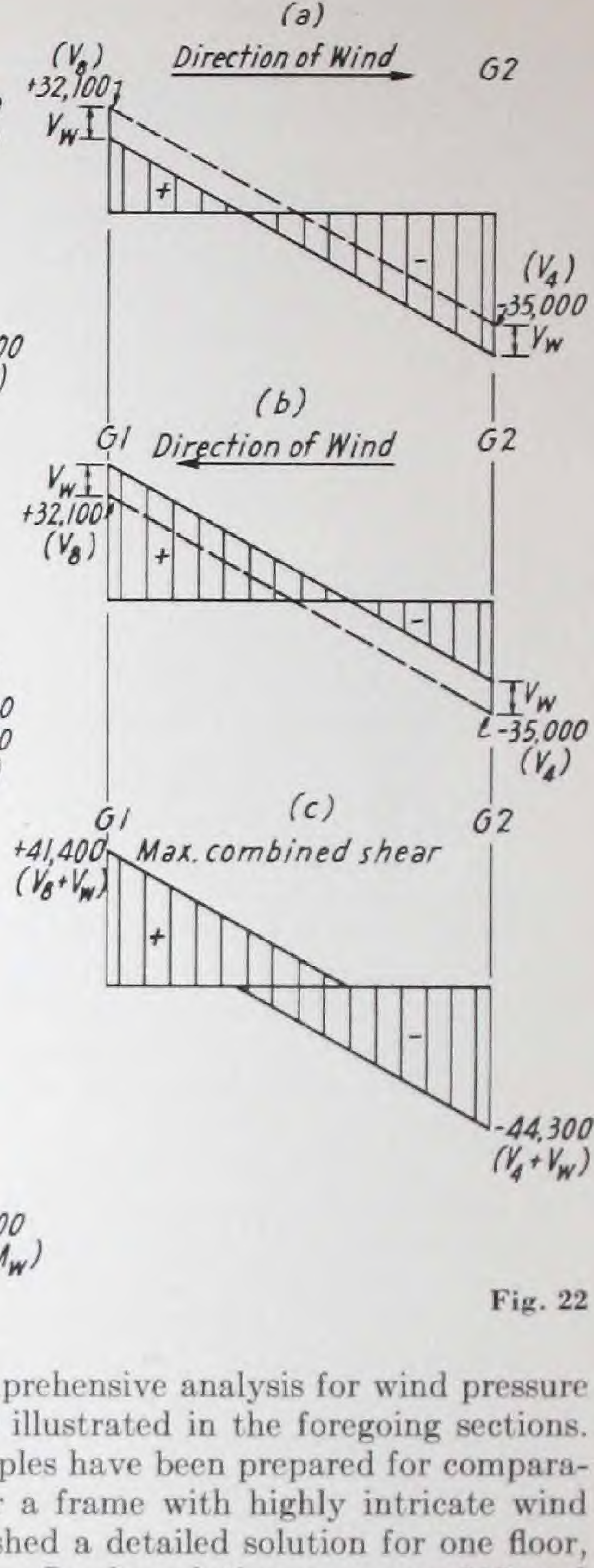

Fig. 22

The principles embodied in a comprehensive analysis for wind pressure have been presented, discussed and illustrated in the foregoing sections. For the sake of simplicity, the examples have been prepared for comparatively simple structural layout. For a frame with highly intricate wind action, Mr. Albert Smith has published a detailed solution for one floor, see Reference 36 in the bibliography. In the solution on pages 919 and 920—it should be observed—the calculations for joists and flush beams might have been lumped in an approximate total without thereby introducing appreciable error in the final moments in the important beams. More than one-half of the calculations given might then have been omitted.

APPENDIX A: DERIVATION OF FORMULAS—VERTICAL LOAD

21. Angle Changes Determined by Moment Area Principle

A prismatic member connecting two joints, A and B, will deflect when loaded as illustrated in Fig. 23. The angles between the line connecting A and B and the tangents to the elastic curve at A and B will be denoted as Θ_A at A and Θ_B at B. The relationship between these angles and the loading may be established by the moment area principle.*

Let $ACDB$ be the moment curve due to the loads, P, placed on the simply supported beam AB. Apply the cross-hatched area—the moment area—as vertical load on a beam similar to AB and determine the end shears, V_A and V_B, produced by this loading. According to the moment area principle,** the angle changes in Fig. 23 are

$$\Theta_A = \frac{1}{EI} \times V_A, \text{ and } \Theta_B = \frac{1}{EI} \times V_B, \qquad (12)$$

in which

E = modulus of elasticity and

I = moment of inertia of the cross section of AB.

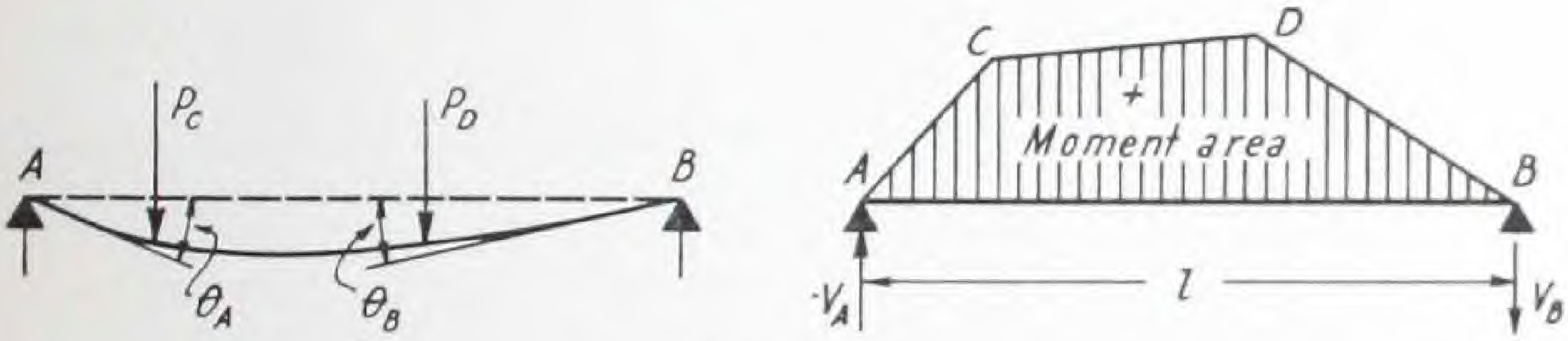

Fig. 23

Equation (12) expresses that "the angle changes at A and B in Fig. 23 equal $(1/EI)$ times the end shears due to the moment area applied as load on beam AB." This is a form of the moment area principle, from which the procedure followed in frame analysis may be developed.

It is customary to define signs for end shear due to vertical loading as positive at the left and negative at the right support, and the same convention applies to angle changes. Further discussion of sign conventions is given in Section 27.

Problem 11. (a) A simply supported beam, AB, has a load, w, uniformly distributed over the entire span length, l. Determine the angle changes at A and B.

*The presentation of the moment area principle dates back to Otto Mohr who, in 1868, established the method of computing deflections as if they were bending moments. The late C. E. Greene at the University of Michigan, about 1873, discovered a related principle of moments of areas. H. Muller-Breslau, in 1885, gave Mohr's principle a general formulation.

**Derivation of the principle is given in many textbooks such as Reference 3 (p. 40) and Reference 14 (p. 216).

The moment curve is a parabola with a maximum ordinate of $\frac{wl^2}{8}$. Load beam AB with the area under the parabola which equals $\frac{2}{3} \times \frac{wl^3}{8}$ and determine the end shears:

$$V_A = -V_B = \tfrac{1}{2} \times \tfrac{2}{3} \times \frac{wl^3}{8} = \frac{wl^3}{24}$$

The angle changes, according to the moment area principle, equal

$$\Theta_A = -\Theta_B = \frac{wl^3}{24EI}. \quad \ldots \ldots \ldots \quad (13)$$

(b) A concentrated load, P, is placed on beam AB at a distance of al from A and bl from B; show that the angle changes due to the load, P, are

$$\Theta_A = \frac{Pabl^2}{6EI}(1 + b), \text{ and } \Theta_B = -\frac{Pabl^2}{6EI}(1 + a). \quad \ldots \quad (14)$$

22. *Angle Changes Due to End Moments*

The loads on beam AB in Fig. 24 are two end moments, M_{AB} and M_{BA}, which deflect the beam as indicated. Let the angles between the line AB and the tangents to the elastic curve at A and B be denoted as Θ'_A at A and Θ'_B at B. The relationship between these angles and the moments that produce them may be established by the moment area principle.

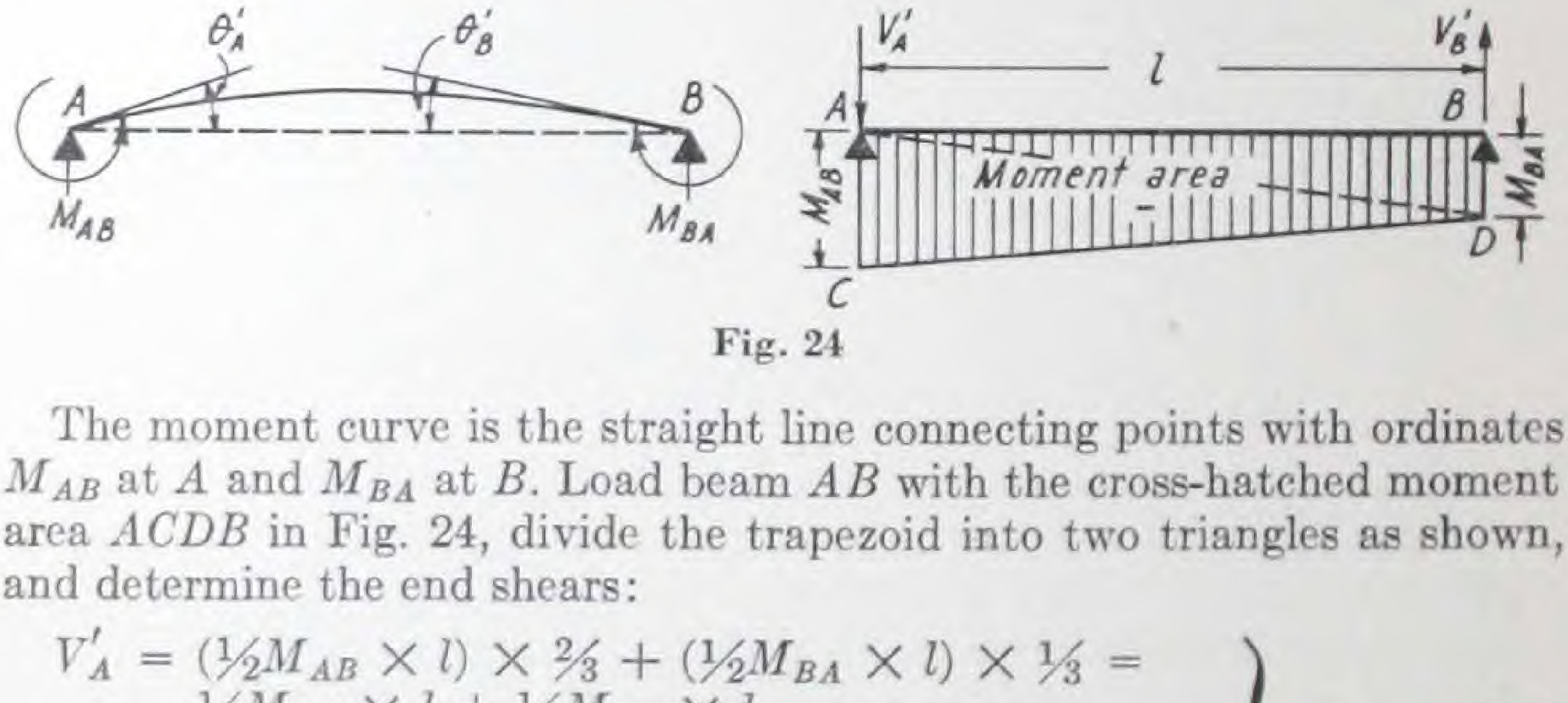

Fig. 24

The moment curve is the straight line connecting points with ordinates M_{AB} at A and M_{BA} at B. Load beam AB with the cross-hatched moment area $ACDB$ in Fig. 24, divide the trapezoid into two triangles as shown, and determine the end shears:

$$\left.\begin{aligned} V'_A &= (\tfrac{1}{2}M_{AB} \times l) \times \tfrac{2}{3} + (\tfrac{1}{2}M_{BA} \times l) \times \tfrac{1}{3} = \\ &\quad \tfrac{1}{3}M_{AB} \times l + \tfrac{1}{6}M_{BA} \times l \\ V'_B &= -(\tfrac{1}{2}M_{AB} \times l) \times \tfrac{1}{3} - (\tfrac{1}{2}M_{BA} \times l) \times \tfrac{2}{3} = \\ &\quad -\tfrac{1}{6}M_{AB} \times l - \tfrac{1}{3}M_{BA} \times l. \end{aligned}\right\} \quad \ldots \quad (15)$$

According to the moment area principle expressed in equation (12), the angle changes Θ' in Fig. 24, derived from equations (15) by setting $\Theta' = \frac{1}{EI} \times V'$, are

$$\left.\begin{aligned} \Theta'_A &= \frac{l}{6EI} \times (2M_{AB} + M_{BA}) \\ \Theta'_B &= -\frac{l}{6EI} \times (2M_{BA} + M_{AB}) \end{aligned}\right\} \quad \ldots \ldots \quad (16)$$

Equations (16) give angle changes in terms of end moments. This relationship is important in frame analysis.

23. *End Moments Required for Fixity*

To determine the fixed end moments, M^F_{AB} and M^F_{BA}, due to a load, P, consider first the beam AB simply supported as in Fig. 23, the tangents being rotated through angles expressed by equations (12). Then apply, as in Fig. 24, two end moments of such magnitude that the tangents at A and B are brought back to their original position. The latter step is equivalent to fixing the two ends of beam AB, and the end moments, M^F_{AB} and M^F_{BA}, equal the moments at the supports of a beam with fixed ends. The procedure is expressed algebraically by the equations $\Theta_A + \Theta'_A = 0$ and $\Theta_B + \Theta'_B = 0$, the Θ-values being taken from equations (12) and (16). The equations become

$$V_A = -\frac{l}{6}(2M^F_{AB} + M^F_{BA})$$

$$V_B = +\frac{l}{6}(2M^F_{BA} + M^F_{AB})$$

Solving for M^F gives

$$\left.\begin{aligned} M^F_{AB} &= -\frac{2}{l}(2V_A + V_B) \\ M^F_{BA} &= +\frac{2}{l}(2V_B + V_A), \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (17)$$

in which V_A and V_B are the end shears due to the moment area for load P applied as vertical load on the simply supported beam AB. Since V_A is positive and V_B is negative, the fixed end moments are negative quantities. Further discussion of sign conventions is given in Section 27.

Equations (17) are used for determination of fixed end moments, such as those presented in Fig. 1.

Problem 12. (a) A beam AB has a load, w, uniformly distributed over the entire span length, l. Determine the moments, M^F_{AB} and M^F_{BA} required to fix the ends of AB.

Inserting in equations (17) the values

$$V_A = -V_B = \frac{wl^3}{24}$$

as determined in Problem 11 gives

$$M^F_{AB} = M^F_{BA} = -\frac{wl^2}{12}. \quad \ldots\ldots\ldots \quad (18)$$

(b) A concentrated load, P, is placed on a beam AB at a distance of al from A and bl from B; show that

$$M^F_{AB} = -ab^2(Pl), \text{ and } M^F_{BA} = -a^2b(Pl). \quad \ldots\ldots \quad (19)$$

Equations (18) and (19) are used frequently in frame analysis; they should be memorized for convenience.

24. *Moments as a Function of End Rotation and Fixed End Moments*

If joints A and B in Fig. 25 are locked artificially before the loads P are applied, fixed end moments—M^F_{AB} and M^F_{BA}—will be created which tend to rotate the joints. When the artificial restraint is removed, the joints will rotate through angles, Θ_A and Θ_B in Fig. 26, until equilibrium is

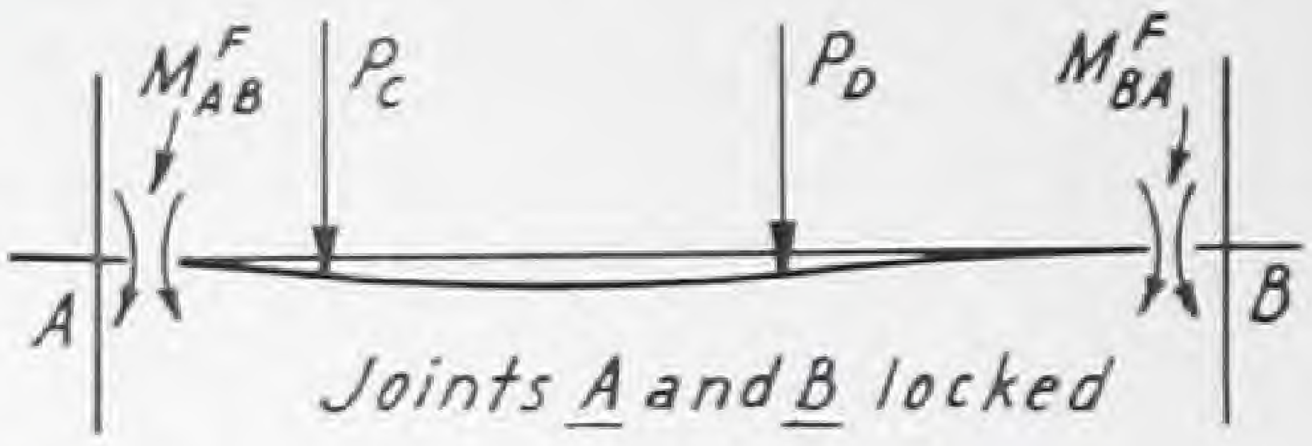

Fig. 25

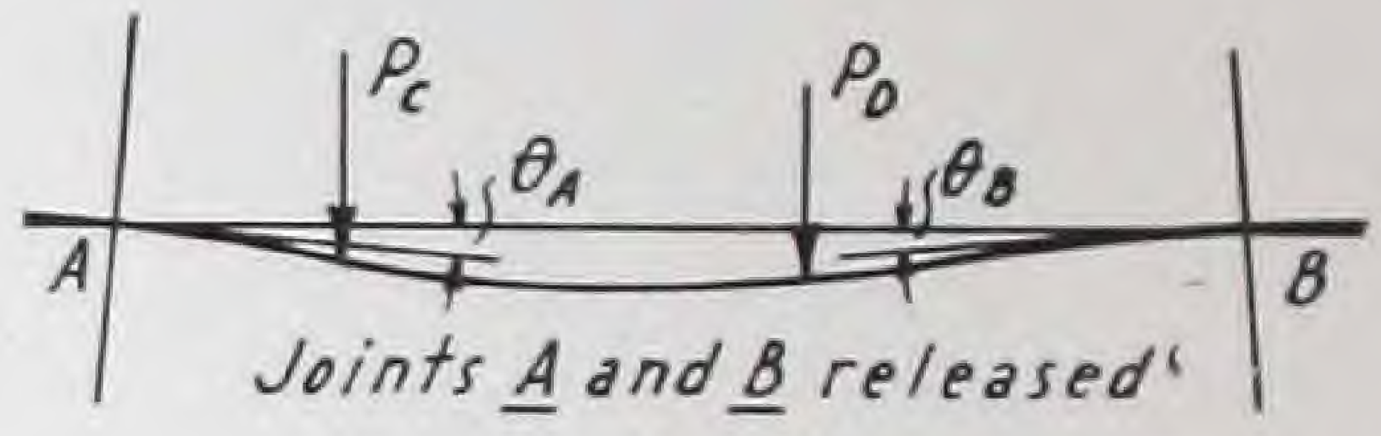

Fig. 26

attained. New end moments, M'_{AB} and M'_{BA}, are thereby induced in AB in addition to the moments M^F that already existed. The final moments are then

$$M_{AB} = M^F_{AB} + M'_{AB}, \text{ and } M_{BA} = M^F_{BA} + M'_{BA} \quad . \quad . \quad . \quad (20)$$

The values of the moments induced by the angle changes may be determined from equations (16) by introducing M' as the moments induced by the angle changes, Θ:

$$2M'_{AB} + M'_{BA} = 6E \times \frac{I}{l} \times \Theta_A$$

$$M'_{AB} + 2M'_{BA} = -6E \times \frac{I}{l} \times \Theta_B .$$

Solving for M'_{AB} and M'_{BA} gives

$$\left.\begin{aligned} M'_{AB} &= 2E \times \frac{I}{l} \times (2\Theta_A + \Theta_B) \\ M'_{BA} &= -2E \times \frac{I}{l} \times (2\Theta_B + \Theta_A), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

which inserted in (20), $\frac{I}{l}$ being denoted as K, gives the final moments

$$\left.\begin{aligned} M_{AB} &= M^F_{AB} + 2 \times (2EK\Theta_A) + 1 \times (2EK\Theta_B) \\ M_{BA} &= M^F_{BA} - 2 \times (2EK\Theta_B) - 1 \times (2EK\Theta_A) \end{aligned}\right\} \quad . \quad . \quad (22)$$

Equations (22) are the Slope-Deflection Equations; they express moments in frames in terms of M^F and joint rotation. Joint translation is disregarded in equations (22) in accordance with common practice.

Equations (22) express that an end moment, M_{AB}, in a beam AB which is part of a frame, may be determined as the algebraic sum of three terms:

(1) fixed end moment at A: M^F_{AB},

(2) moment induced by joint rotation at the *near* end, A: $2(2EK\Theta_A)$, and

(3) moment induced by joint rotation at the *far* end, B: $1(2EK\Theta_B)$.

Fixed end moments may be determined when loads and spans are given. The values of $EK\Theta$ used in (22) will be given further study.

25. *Joint Rotation a Function of Stiffness*

Fig. 27 shows four members which are part of a frame and intersect at joint A. Let joint A be rotated through an angle, Θ_A, by a moment, U_A, applied at A. At the far ends of the members, four different conditions are assumed for illustration: joint B is fixed ($\Theta_B = 0$); at joint C the rotation is $-\Theta_A$, at D it is $+\Theta_A$, and at F it is $f\Theta_A$, the value of f depending upon the conditions at F. The $\frac{I}{l}$-values will be denoted as K_{AB}, K_{AC}, K_{AD} and K_{AF}, which represent the stiffness of the various members.

The joint rotations in Fig. 27 have induced end moments which may be expressed by use of equations (21). Since joint A is in equilibrium, the sum of the end moments at A must equal U_A. This requirement is expressed in the summation that follows:*

$$
\begin{array}{lll}
M_{AB} = +2EK_{AB} \times (2\Theta_A + (0)) & = +[2EK_{AB}(2\Theta_A)] \\
M_{AC} = +2EK_{AC} \times (2\Theta_A + (-\Theta_A)) & = +[2EK_{AC}(\Theta_A)] \\
M_{AD} = -2EK_{AD} \times (2\Theta_A + (+\Theta_A)) & = -[2EK_{AD}(3\Theta_A)] \\
M_{AF} = -2EK_{AF} \times (2\Theta_A + (f\Theta_A)) & = -[2EK_{AF}(2 + f)\Theta_A] \\
\hline
\Sigma M_{AX} = U_A = 2E\Theta_A \times [2K_{AB} + K_{AC} + 3K_{AD} + K_{AF}(2 + f)]. & &
\end{array}
$$

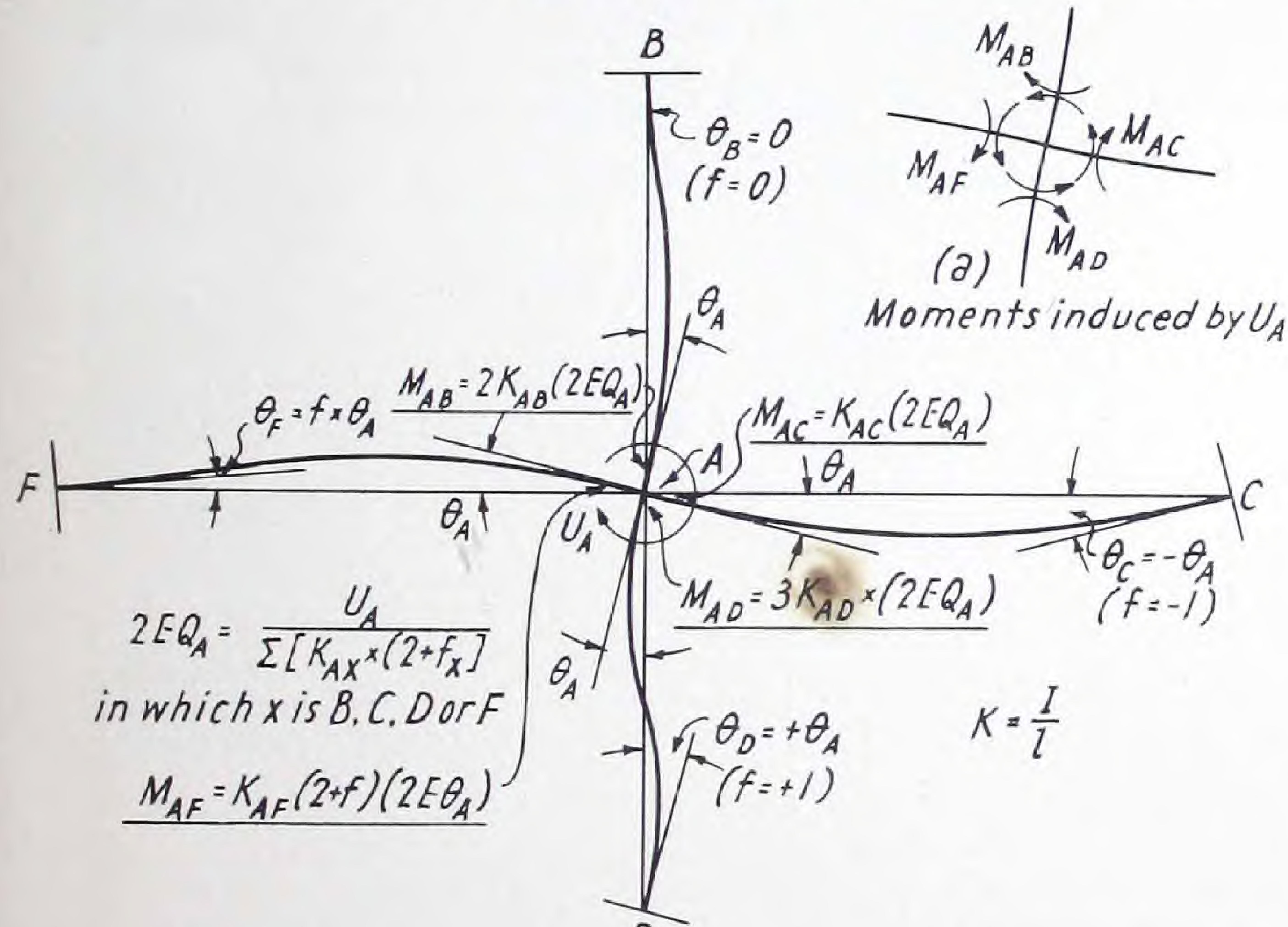

Fig. 27

The four moments induced by U_A are shown diagrammatically in Fig. 27(a). Since all four moments tend to rotate joint A in the same direction, it is the sum of the quantities within the four brackets above the summation line that equals U_A.

*For discussion of signs used in the derivation of equation (23), see Section 27.

The equation $\Sigma M_{AX} = U_A$ gives

$$2E\Theta_A = \frac{U_A}{2K_{AB} + K_{AC} + 3K_{AD} + K_{AF}(2+f)}. \quad . \quad . \quad (23)$$

Equation (23) expresses that $2E\Theta_A$ equals U_A (unbalanced moment due to the fixed end moments at A, see definition in Section 27), divided by the sum of the products $K \times (2 + f)$ written for each adjacent member. By comparing equation (23) and Fig. 27, it is seen that

$2 + f = 2$, when angle change at far end equals 0: $f = 0$
$2 + f = 1$, when angle change at far end equals $-\Theta_A$: $f = -1$
$2 + f = 3$, when angle change at far end equals $+\Theta_A$: $f = +1$

Equation (23), written in its general form, becomes

$$2E\Theta_A = \frac{U_A}{\Sigma[K_{AX}(2 + f_X)]}.^{*} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

26. Derivation of Formulas for Joint Coefficient, Q

A quantity denoted as Q is introduced and defined in Fig. 2. It will be seen by comparing Fig. 2 with equations (22) that the value of Q_A is

$$Q_A = \frac{2EK_{AB}\Theta_A}{U_A}$$

An expression for the right side of the equation is obtained by multiplying with $\frac{K_{AB}}{U_A}$ in equation (24), which gives

$$Q_A = \frac{2EK_{AB}\Theta_A}{U_A} = \frac{K_{AB}}{\Sigma[K_{AX}(2 + f_X)]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Therefore, Q_A is a function of
K_{AX}: the I/l-values of the members intersecting at joint A, and
f_X: the degree or type of restraint at the *far* end of the members.

An approximation is now introduced by inserting in equation (25):

$2 + f_X = 2$ for all columns,
$2 + f_X = 2$ for beam adjacent to exterior joints.
$2 + f_X = 1$ for beams adjacent to interior joints.

If, in addition, the stiffness of beams adjacent to any one joint is set equal to K and the stiffness of columns at the same joint equal to nK, equation (25) may be written in the simple form:

$$Q = \frac{K}{2(2nK) + 2K} = \frac{1}{4n + 2}.^{**} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Either equation (25) or (26) may be used to determine Q. The procedure in Steps (1) to (6) in Section 2 would give mathematically exact results

*A quite accurate and very quick analysis may be made by designers who are accustomed to judging the *relative* magnitude of joint rotations. The procedure is simply to estimate the value of f for the far ends, determine joint rotations from (24), and compute the final moments from equations (22).

Professor Maney has developed an approximate method involving the following steps: (1) at alternate joints set $f_X = 0$ and calculate Θ from equation (24); (2) using these Θ-values, calculate Θ at the intermediate joints; (3) re-calculate Θ at joints in first group; (4) repeat steps (2) and (3) if necessary; (5) calculate moments by equations (22). This method was presented by John E. Goldberg, see Reference 9. A similar procedure has been presented in a manuscript by Raymon C. Buell.

**For roofs, the equation becomes $Q = \frac{1}{2n + 2}$, based on assumptions similar to those made for floors.

if f in (25) were known; it becomes an approximate method* because f is estimated or approximated as in (26).

The value of f appears always together with K in the product of $K(2 + f)$. The K-values are not known beforehand, but must be estimated before a frame can be analyzed; it is then reasonable to estimate f also and to use the simplified equation (26). When reviewing a frame with given proportions, the "physical characteristics of the structure make it impossible to determine the actual bending moment to a greater degree of accuracy than ± 5 per cent."** Therefore, even when proportions are given, a relatively poor estimate of f is of little consequence.

27. *Conventions and Use of Signs*

Sign conventions used in current practice and their application to the derivations in the foregoing analysis will be discussed.

A beam, AB, through which an arbitrary vertical section S-S is drawn, dividing the beam into two parts "A" and "B," is shown diagrammatically in Fig. 28(a). Remove "B" and add to "A" the internal stresses, C, T and V, exerted by "B." The stresses C and T form a couple, the internal moment; since "A" is in equilibrium, the internal moment equals the couple formed by the loads and reactions on "A," the external moment. Transposing the procedure by removing "A" and adding in S-S the internal stresses exerted upon "B" gives an internal moment which is numerically equal to the internal moment on "A" but opposite in direction.

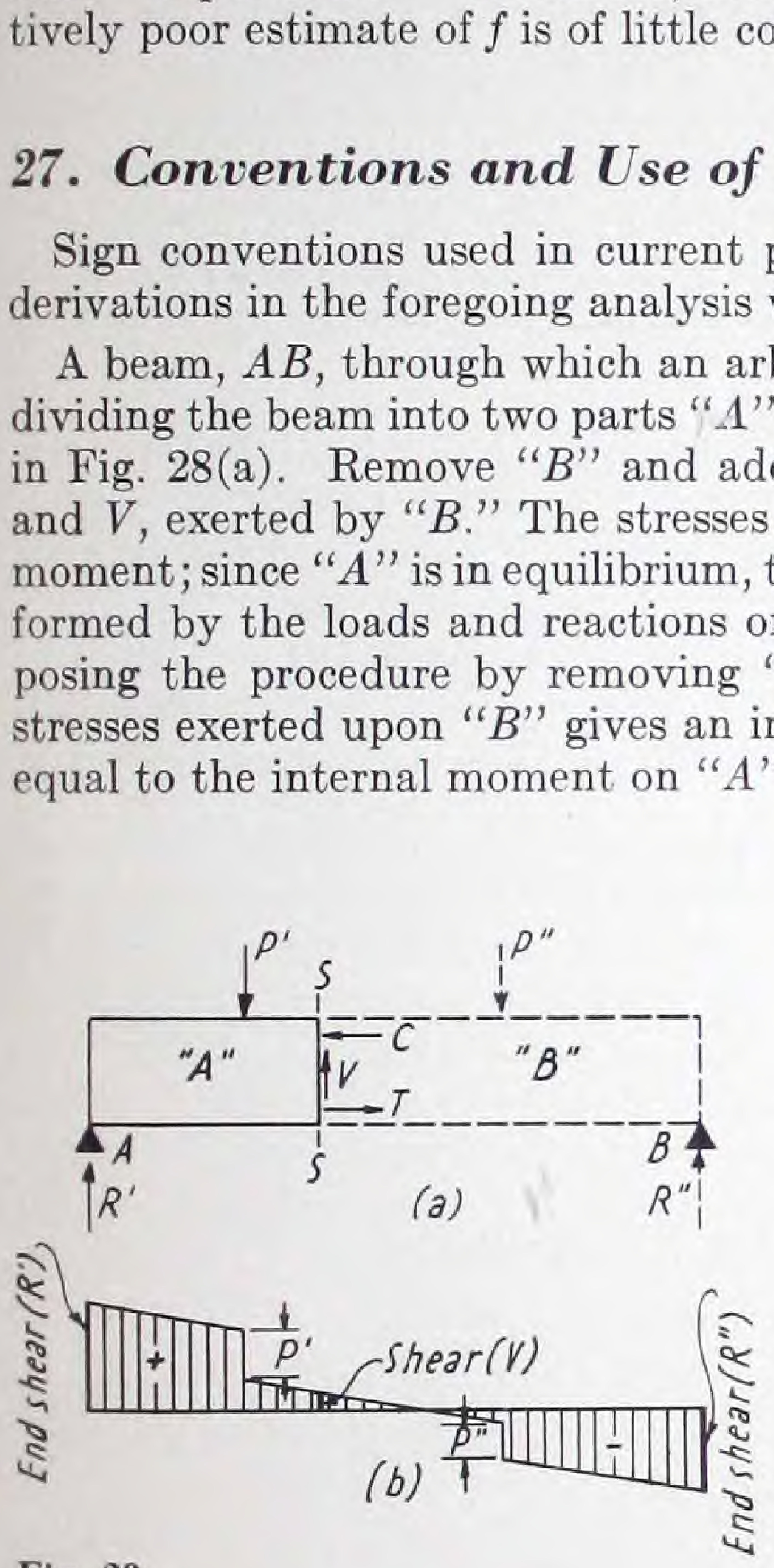

Fig. 28

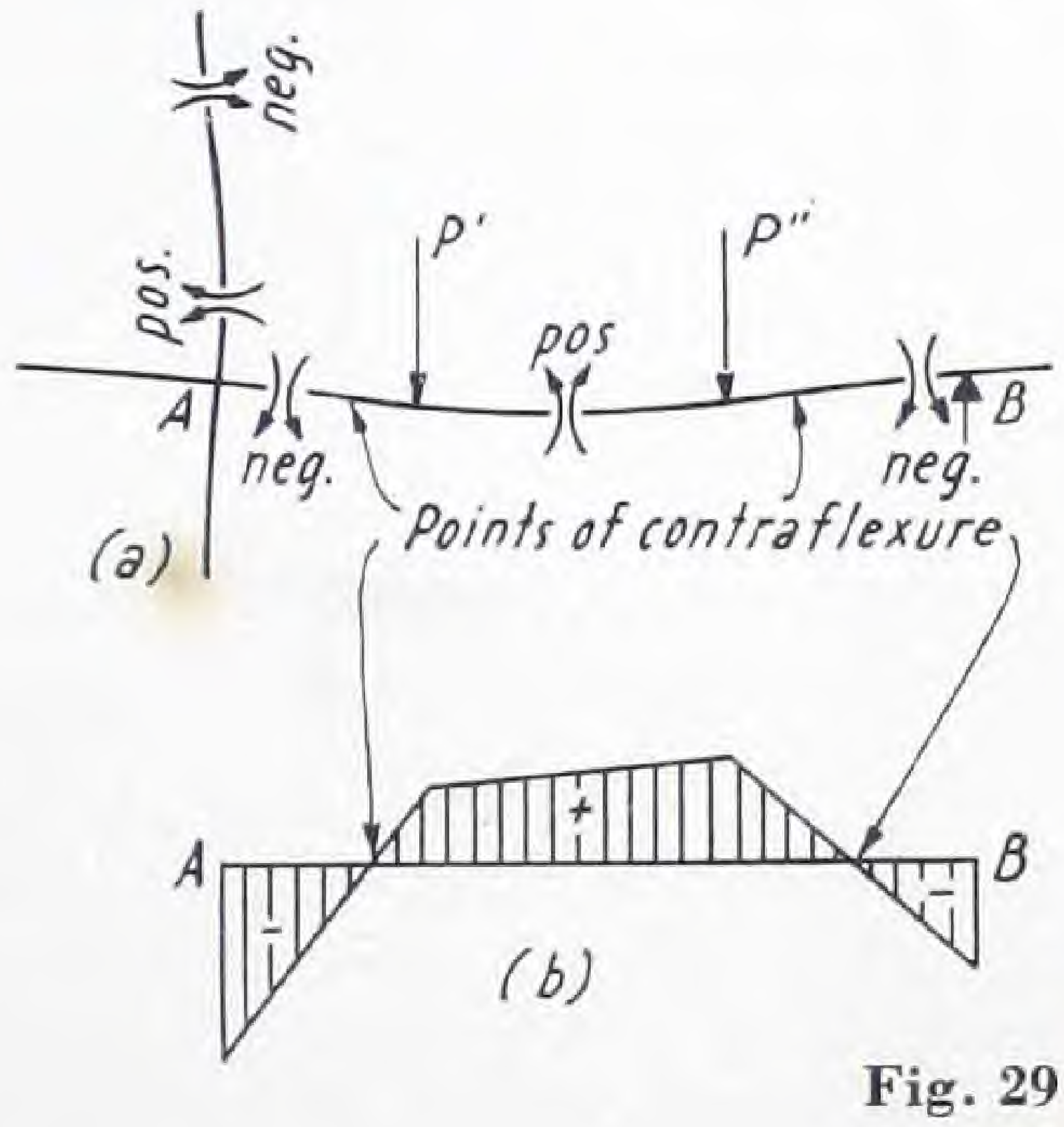

Fig. 29

The internal moments at any section may be indicated by *two* curved arrows as in Fig. 29(a). If the section is taken immediately outside joint A in Fig. 29, the arrow which is closer to A indicates the moment with which beam AB tends to rotate joint A, and the other arrow indicates the moment exerted by the joint upon the beam.

*Various analytical procedures have been proposed in which a relative stiffness such as $K \times (2 + f)$ is introduced. See, for example, Professor L. E. Grinter's discussion of "A Direct Method of Moment Distribution," *Proc. A.S.C.E.*, March, 1935, page 426.
**Professor L. E. Grinter in *Proc. A.S.C.E.*, March, 1935, page 428.

The internal force V, the shear, in section S-S in Fig. 28(a) is defined as the algebraic sum of the reaction, R', and the load, P', on the part of beam AB to the left of S-S. The common convention is to take the shear at the end of the beam as positive at the left support as in Fig. 28(b); the end shear is then negative at the right support. Apply this definition to the type of end shear treated in equations (12) and (16); for a deflection as in Fig. 23, Θ becomes positive at A and negative at B, and these signs are reversed for a deflection as in Fig. 24. From the definitions of shear, it follows that fixed end moments are negative (see discussion of equation (17) in Section 23). The sign conventions for shear and moment are interlocked by the moment area principle expressed in equation (12). Since fixed end moments for vertical loading are negative and occur where the elastic curve is humped, "a moment is positive when it sags the beam;" this will be referred to as sign convention number (1). For columns, the same sign convention may be applied if the sheet is turned to read from the right. Positive moments will then create tension in the bottom of beams and in the right hand side of columns.

In dealing with joint rotation, it is convenient to adopt another convention, number (2): a moment exerted upon a joint by a beam is positive when it tends to rotate the joint in the clockwise direction.* Convention (2) would make all the moments at joint A in Fig. 27(a) negative. This sign convention is preferred in analysis, the determination of rotation and moments. The ultimate purpose of analysis is design, or determination of stresses; and for this purpose it is customary to use sign convention (1).

Fig. 29 shows deflected members in part of a frame, and internal moments are indicated by double arrows. The moment equals zero at the two points in AB where the curvature changes from a sag to a hump, the points of contraflexure. The moments, positive between and negative outside these points, according to convention (1), may be plotted as indicated in Fig. 29(b). If convention (2) were adopted, the moment in Fig. 29(a) would become positive at A but negative at B; this would be confusing in plotting moment curves. Convention (1) is adopted in this text, although certain derivations may appear less direct; but the application of formulas and procedure has doubtlessly been simplified.

The numerical value of U_A is defined as the algebraic difference between the fixed end moments at A. According to equation (23), U has the same sign as Θ. This requirement will be fulfilled in Fig. 2 when U at any point is taken as the algebraic difference "M^F to the left minus M^F to the right of the joint," the fixed end moments being used with their proper signs. If at joint A, for example, M^F_{AB} is numerically larger than M^F_{AC}, then $U_A = M^F_{AC} - M^F_{AB}$ is positive, and Θ also becomes positive; if M^F_{AB} is smaller than M^F_{AC}, both U_A and Θ_A are negative.

*See Reference 1, page 14.

28. Formulas for Moments in Columns

Let K_c and K_b be the I/l-ratio for columns and beams in Fig. 8(a) and set $n = K_c/K_b$. The deflection of the columns in Fig. 8(a) is similar to that of member AD in Fig. 27 for which $f = +1$; therefore, $K_c(2 + f) = 3K_c$. The beams in Fig. 8(a) deflect as member AC in Fig. 27 for which $f = -1$, and $K_b(2 + f) = K_b$. Insert $3K_c$ and K_b in the equation in Fig. 27 which gives the relationship between Θ and U in Fig. 8(a):

$$2E\Theta = \frac{U}{2(3K_c) + 2(K_b)}. \qquad (27)$$

The end moment, M, induced by joint rotations as in member AD in Fig. 27 is

$$M = 3K_c(2E\Theta),$$

which combined with equation (27) gives

$$M = K_c\frac{3U}{6K_c + 2K_b} = \frac{3n}{6n + 2} \times U. \qquad (28)$$

At exterior columns, with loading arranged as in Fig. 8(c) to give approximately maximum column moment, the value of $K_c(2 + f)$ equals $3K_c$ as for interior columns; but the shape of the deflected beam in the end span is somewhere between the shapes of $AB(f = 0)$ and $AC(f = -1)$ in Fig. 27. Adopting $f = -0.5$ gives $K_b(2 + f) = 1.5K_b$, and equations (27) and (28) become modified as follows:

$$2E\Theta = \frac{U}{2(3K_c) + 1(1.5K_b)} \qquad (29)$$

and

$$M = \frac{2n}{4n + 1} \times U. \qquad (30)$$

APPENDIX *B*: DERIVATION OF FORMULAS—WIND PRESSURE

29. Assumptions and Distribution of Wind Pressure

Let A, B, C, D and F in Fig. 30 be joints in a bent which is deformed by bending due to wind pressure. While investigating the conditions around joint A, make the following assumptions:*

(a) joints F, A and C lie on a straight line

(b) joints B, A and D lie on a straight line

(c) the same angle change, Θ_A, exists at F, A and C, Θ_A being measured from a horizontal line

(d) the same angle change, Θ_A, exists at B, A and D, Θ_A being measured from vertical lines.

*Suggested by Professors Wilson and Maney on page 25 in Reference 21.

The assumptions, incorporated in Fig. 30, place the point of contraflexure at the midpoints of all members. They also provide for a method of distribution of wind pressure as derived in the paragraphs which follow.

The part of the bent shown diagrammatically in Fig. 30 will deflect under wind pressure, Θ being the angle change at the joints and R an angle representing translation of the joints.

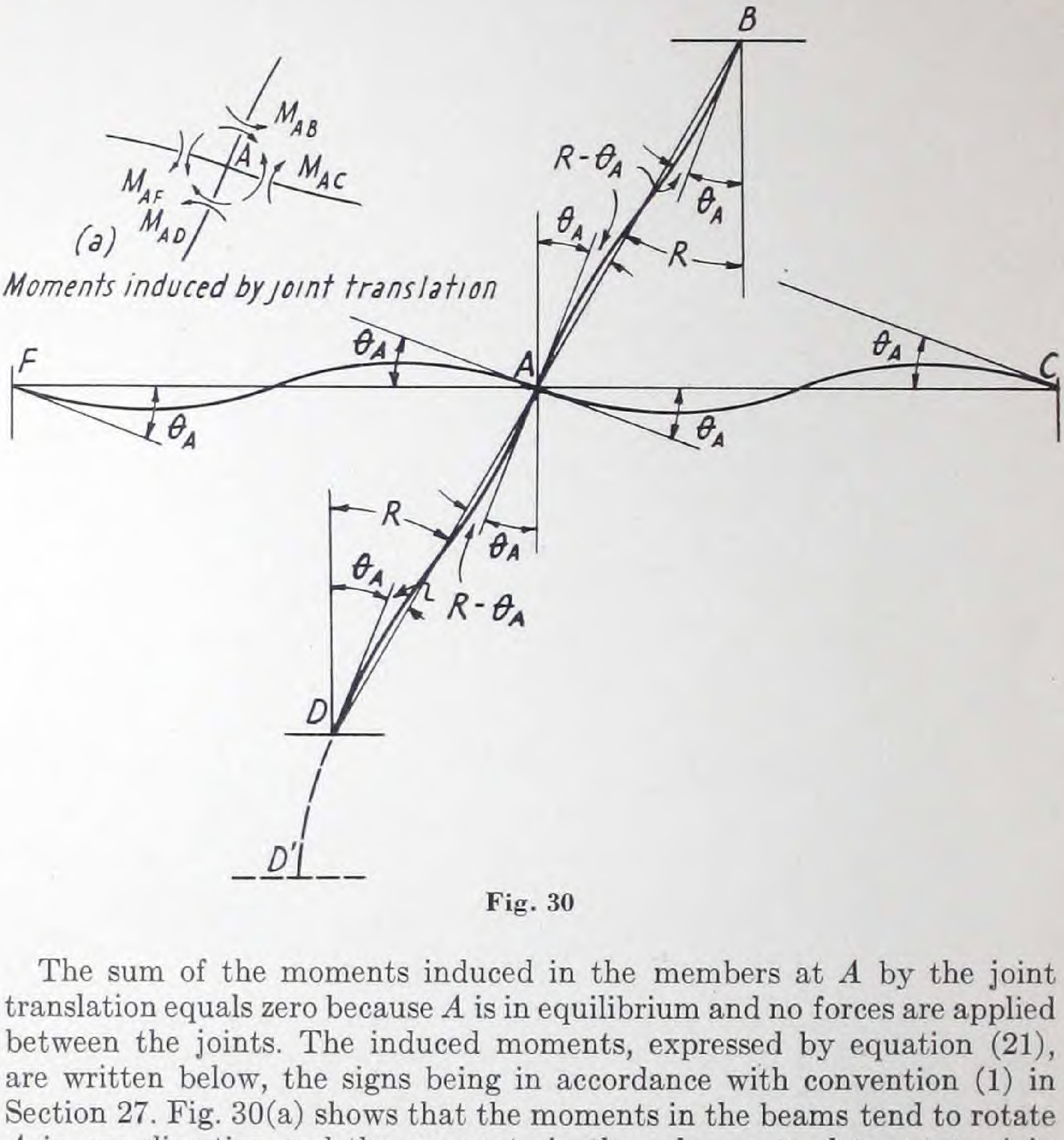

Fig. 30

The sum of the moments induced in the members at A by the joint translation equals zero because A is in equilibrium and no forces are applied between the joints. The induced moments, expressed by equation (21), are written below, the signs being in accordance with convention (1) in Section 27. Fig. 30(a) shows that the moments in the beams tend to rotate A in one direction and the moments in the columns tend to rotate A in the opposite direction. It is, therefore, the sum of the four moments within the parentheses that equals zero.

$$M_{AC} = +2EK_{AC}\times[2\Theta_A+(+\Theta_A)] = +(6EK_{AC}\times\Theta_A)$$
$$M_{AF} = -2EK_{AF}\times[2\Theta_A+(+\Theta_A)] = -(6EK_{AF}\times\Theta_A)$$
$$M_{AB} = +2EK_{AB}\times[-3(R-\Theta_A)] = +(6EK_{AB}\times\Theta_A-6EK_{AB}\times R)$$
$$M_{AD} = -2EK_{AD}\times[-3(R-\Theta_A)] = -(6EK_{AD}\times\Theta_A-6EK_{AD}\times R)$$
$$\Sigma M_{AX} = 0 = 6E\Theta_A\times\Sigma(K_{AX})-6ER\times(K_{AB}+K_{AD}).$$

The equation below the summation line reduces to

$$0 = \Theta_A \times (\Sigma K_{AX}) - R(K_{AB} + K_{AD}),$$

from which

$$\Theta_A = R \times \frac{K_{AB} + K_{AD}}{\Sigma K_{AX}}. \quad (31)$$

Combine the equation

$$R - \Theta_A = R \times \frac{(\Sigma K_{AX}) - (K_{AB} + K_{AD})}{\Sigma K_{AX}} = R \times \frac{K_{AC} + K_{AF}}{\Sigma K_{AX}}$$

with

$$M_{AB} = -6EK_{AB}(R - \Theta_A),$$

which gives

$$M_{AB} = -6ER \times K_{AB}\frac{K_{AC} + K_{AF}}{\Sigma K_{AX}}. \quad (32)$$

If the shear in column AB is denoted as V_A, then

$$V_A = \frac{2}{h} \times M_{AB} = -\frac{12ER}{h} \times K_{AB}\left(\frac{K_{AC} + K_{AF}}{\Sigma K_{AX}}\right), \quad (33)$$

in which the numerator within the parenthesis is the sum of the K-values for the beams at A and the denominator is the sum of the K-values for all members at A.

The value of R, the only unknown in equations (31), (32) and (33), need not be determined when R/h is constant for all columns in a story. In this case, the *relative* values, v_A, of the shears, V_A, taken by each column become equal to

$$v_A = K_{AB}\frac{K_{AC} + K_{AF}}{\Sigma K_{AX}}^{*}. \quad (34)$$

In addition, the sum of all column shears in a story equals the total wind shear. From this requirement together with equation (34), shears and subsequently moments may be calculated in the columns. Shears and moments may then be computed in the beams by using the assumption of contraflexure at midspan.

Let the dotted line extending downward past D in Fig. 30 indicate that AD' is a basement story column, with height h' and assumed fixed at the footing, D'. The regular distribution of wind pressure may be applied in this case also, except that for columns as AD', the effective story height, h, should be taken somewhat smaller than the actual story height h', say, $h = \frac{3}{4} \times h'$.

*Equation (34) gives shear in column AB above joint A. For the column below, AD, substitute K_{AD} for K_{AB} in equation (34).

BIBLIOGRAPHY

Vertical Load Analysis

1. "Analysis of Statically Indeterminate Structures by the Slope-Deflection Method," W. M. Wilson, F. E. Richart and Camillo Weiss, *Bulletin 108*, Univ. of Ill., Eng. Exp. Sta., 1918.

2. "Reinforced Concrete Design," Oscar Faber, published by Arnold, London, Vol. 2, 1924.

3. "Statically Indeterminate Stresses," Parcel and Maney, published by John Wiley & Sons, 1926.

4. "Moments in Restrained and Continuous Beams by the Method of Conjugate Points," L. H. Nishkian and D. B. Steinman, *Trans. A.S.C.E.*, 1927, Vol. 90, pp. 1-206 (incl. discussion).

5. "Continuity as a Factor in Reinforced Concrete Design," Hardy Cross, *Proc. A.C.I.*, 1929, pp. 669-711.

6. "Simplified Rigid Frame Design," Hardy Cross, *Proc. A.C.I.*, Vol. 26, 1930, pp. 170-183.

7. "Standards of Design for Concrete," U. S. Navy Dept., Bureau of Yards and Docks, U. S. Government Printing Office, Washington, 1930.

8. "The Column Analogy," Hardy Cross, *Bulletin 215*, Univ. of Ill., Eng. Exp. Sta., 1930.

9. "Vertical-Load Analysis of Rigid Building Frames Made Practicable," John E. Goldberg, *Engineering News-Record*, November 12, 1931, pp. 770-772.

10. "Analysis of Continuous Frames by Distributing Fixed End Moments," Hardy Cross, *Trans. A.S.C.E.*, 1932, Vol. 96, pp. 1-156 (incl. discussion).

11. "Continuous Frames of Reinforced Concrete," Cross and Morgan, published by John Wiley & Sons, 1932.

12. "The Modified Slope-Deflection Equations," L. T. Evans, *Proc. A.C.I.*, 1932, pp. 109-130.

13. "Design of Continuous Frames Having Variable Moment of Inertia," Thor Germundsson, *Civil Engineering*, October, 1932, pp. 647-648.

14. "The Design of Steel Mill Buildings and the Calculation of Stresses in Framed Structures," Milo S. Ketchum, published by McGraw-Hill, 1932.

15. "Continuous-Beam Analysis by Direct Distribution," J. C. Schulze and W. A. Rose, *Engineering News-Record*, December 20, 1934, pp. 782-785.

16. "Analysis of Continuous Structures by Traversing the Elastic Curves," Ralph W. Stewart, *Proc. A.S.C.E.*, October, 1934, pp. 1125-1134.

17. "Handbook of Rigid Frame Analysis," L. T. Evans, printed by Edwards Brothers, Inc., Ann Arbor, Mich., 1934.

18. "Analysis of Continuous Frames by the Method of Restraining Stiffnesses," by E. B. Russell, published by Ellison and Russell, San Francisco, Calif., 1934.

19. "A Direct Method of Moment Distribution," T. Y. Lin, *Proc. A.S.C.E.*, December, 1934, pp. 1451-1461. (See discussion by L. E. Grinter, March, 1935, p. 425.)

20. "Rigid-Frame Design in the Bureau of Yards and Docks," G. A. Hunt, *Engineering News-Record*, June 27, 1935, pp. 915-917.

BIBLIOGRAPHY

Wind Stress Analysis

21. "Wind Stresses in the Steel Frames of Office Buildings," W. M. Wilson and G. A. Maney, *Bulletin 80*, Univ. of Ill., Eng. Exp. Sta., 1915.

22. "Wind Stresses in the Steel Frames of Office Buildings," Albert Smith and W. M. Wilson, *Jl. Western Soc. of Eng.*, Apr., 1915, pp. 341-390.

23. "Column Deflection Method for Designing Lateral Bracing," John C. Van der Mey and Felix H. Spitzer, *Engineering News-Record*, January 19, 1928, pp. 106-108.

24. "Design of Tall Building Frames to Resist Wind," Morris and Rose, *Bulletin 48*, Ohio State Univ., 1929.

25. "Wind Stresses in Tall Buildings," R. Fleming, *Engineering and Contracting*, June, 1929, pp. 250-252.

26. "Wind Stresses in Buildings," R. Fleming, published by John Wiley & Sons, 1930.

27. "Wind on Tall Buildings," E. E. Seelye, *Engineering News-Record*, March 20, 1930, pp. 481-483.

28. "Wind Bracing: The Importance of Rigidity in High Towers," H. V. Spurr, published by McGraw-Hill, 1930.

29. Expert Testimony on Wind Design for Tall Buildings:

    I. "A Review of Current Theories," D. C. Coyle, *Engineering News-Record*, June 4, 1931, pp. 932-934.

    II. "Assumed Loads and Fiber Stresses," Albert Smith, *Engineering News-Record*, June 11, 1931, pp. 971-974.

    III. "Chord Deflections Control Web System Design," Henry V. Spurr, *Engineering News-Record*, June 18, 1931, pp. 1012-1014.

    IV. "An Analysis of Some Basic Assumptions," C. T. Morris, *Engineering News-Record*, June 25, 1931, pp. 1050-1051.

30. "Wind Stresses on Portal Frames," W. S. Gray, *Concrete and Constr. Eng.*, 1932, pp. 623-631.

31. "Deflections and Vibrations in High Buildings," L. J. Mensch, *Proc. A.C.I.*, Vol. 28, 1932, pp. 387-404.

32. "Wind-Stress Analysis and Moment Distribution," R. Fleming, *Engineering News-Record*, February 9, 1933, pp. 194-195.

33. "Wind-Bracing Problems," A. Smith, *Jl. Western Soc. of Eng.*, February, 1933, pp. 1-18.

34. "Wind Stresses by Slope Deflection and Converging Approximations," J. E. Goldberg, *Trans. A.S.C.E.*, 1934, pp. 962-985.

35. "Wind Stress Analysis Simplified," L. E. Grinter, *Trans. A.S.C.E.*, 1934, pp. 610-669. Discussion closed in *Proc. A.S.C.E.*, January, 1934, pp. 93-102.

36. "Basis of Design for Hurricane Exposure," Albert Smith, *Proc. A.C.I.*, Vol. 27, 1931, pp. 903-924.

37. "Structural Frameworks," Thomas F. Hickerson, The University of North Carolina Press, 1934.

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